

Chip-Firing Games on Graphs and Proof Formalization

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Budapest Semesters in Mathematics

Interactive demo: <https://npflueger.github.io/chipfiring/>

Lean4 Docs: <https://dhyeymavani.com/chip-firing-with-lean/docs/ChipFiringWithLean/RiemannRochForGraphs.html>

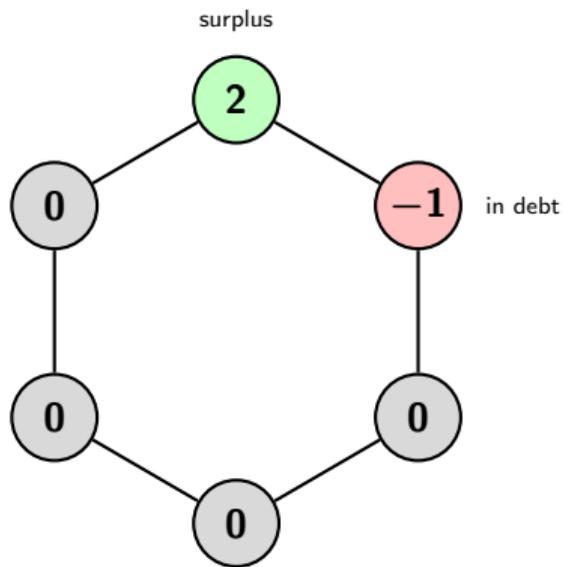
It is great to be back!



Me in Budapest, Fall 2007, excited to find courses with graphs in them!

A Curious Economy

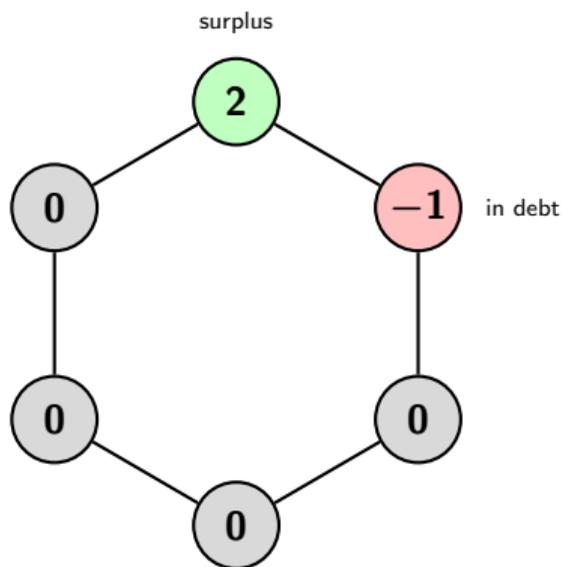
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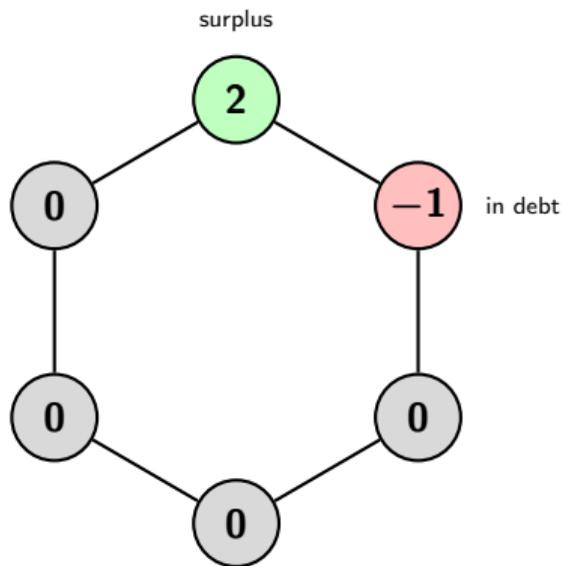
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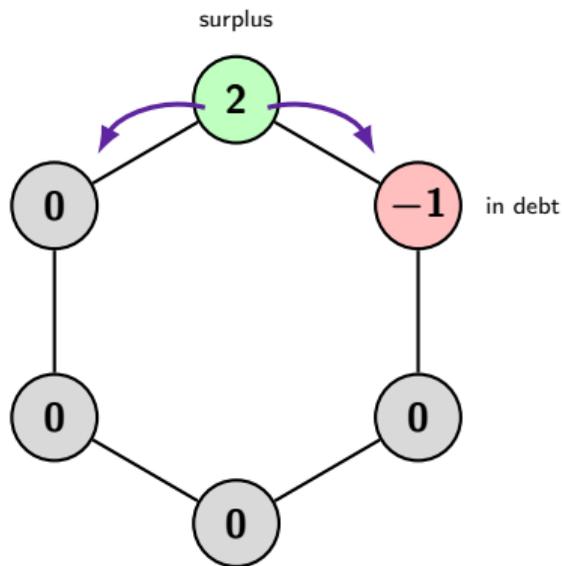
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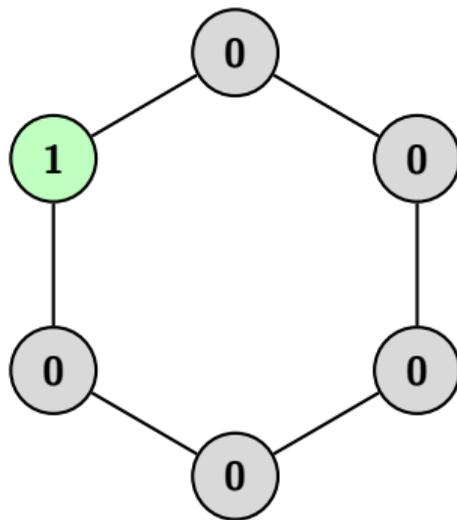
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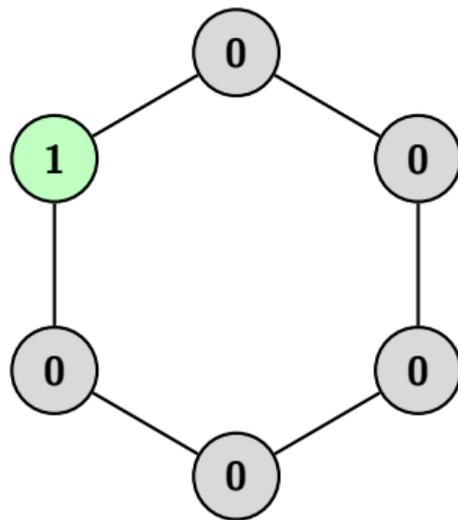
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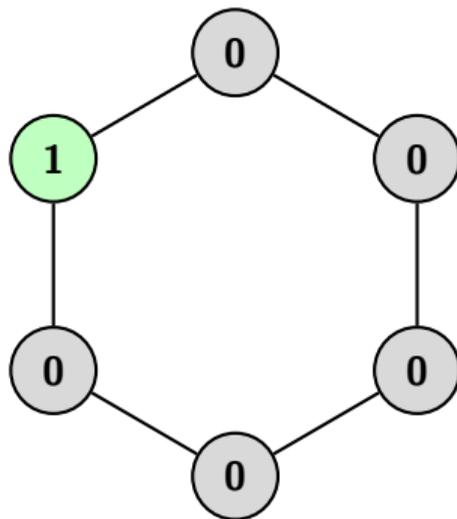
Can everyone be brought out of debt?

If so, the configuration is **winnable**.

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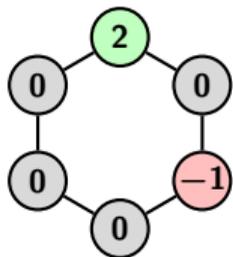
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(Later, we'll discuss two more complex games: the *gonality game* and the *rank game*.)

Examples on C_6

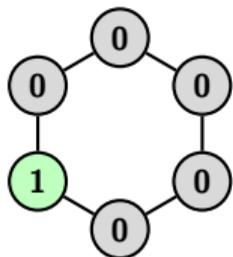
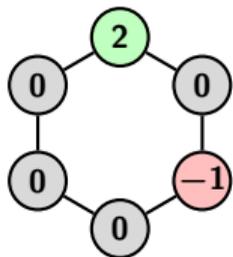
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Not winnable

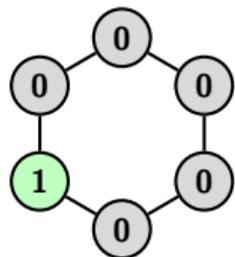
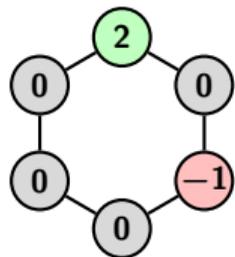
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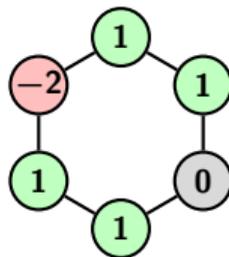
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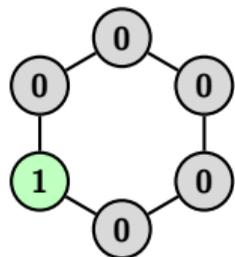
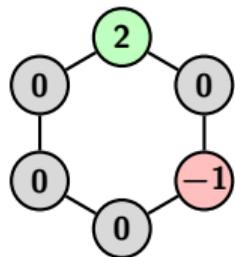


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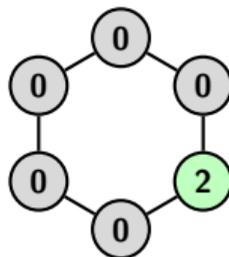
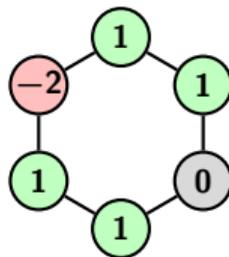


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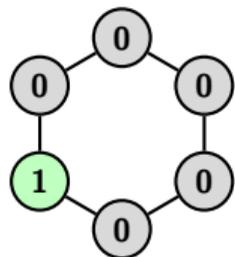
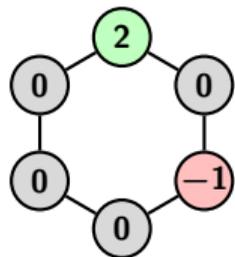
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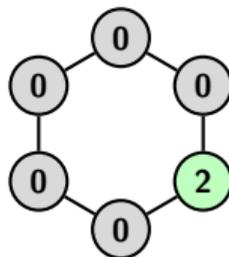
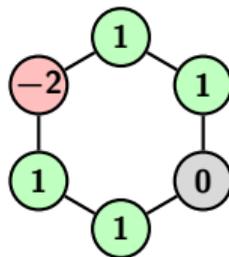
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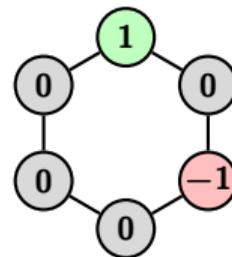
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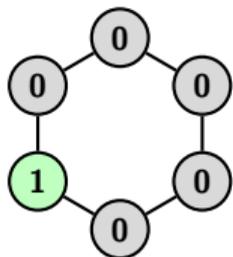
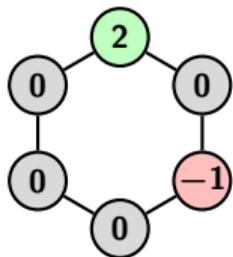


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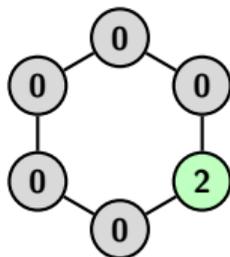
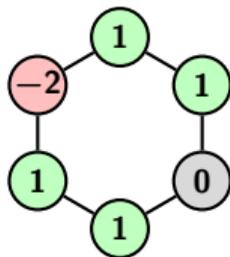


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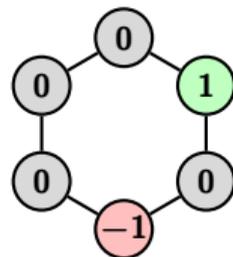
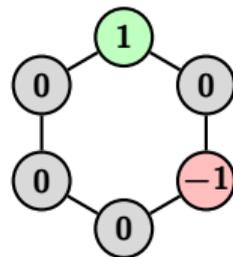
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Winnability on cycles

If you try examples like these, you might notice this pattern:

Theorem

On a cycle graph C_n , any configuration with ≥ 1 total forints is winnable.

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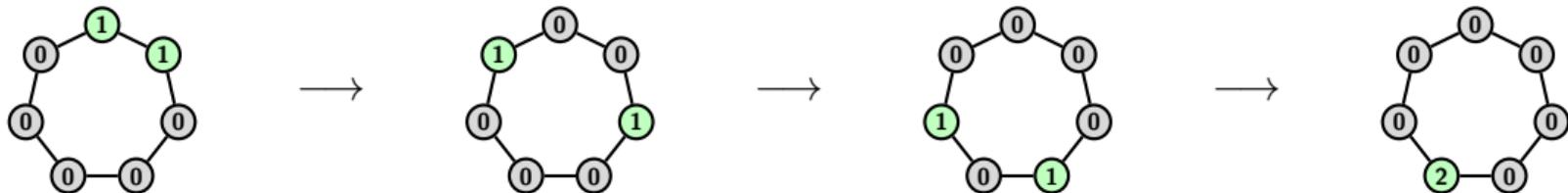
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Proof sketch.

Choose one forint as a “balancer.” We claim that any other forint can be moved to any desired vertex if we also move the balancer in a mirrored way. In this way, use positive forints one by one to balance negative forints. □



Winnability on cycles

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Definition

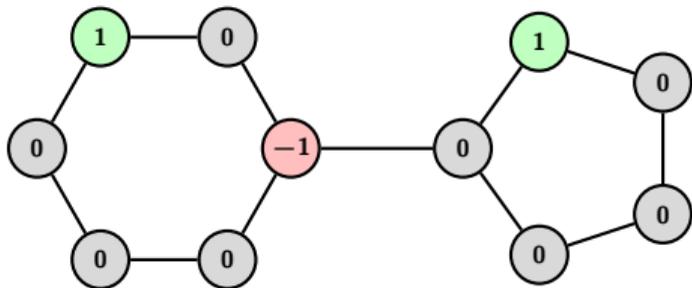
The *degree* of a chip configuration is the total number of forints (chips) it has.

Question

Given a graph G , what is the minimum degree d such that every configuration of degree $\geq d$ is winnable?

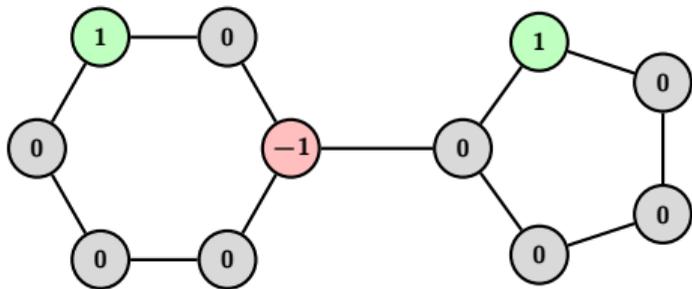
A barbell graph

Consider this *barbell graph*, joining C_6 to C_5 with a “bridge” edge.



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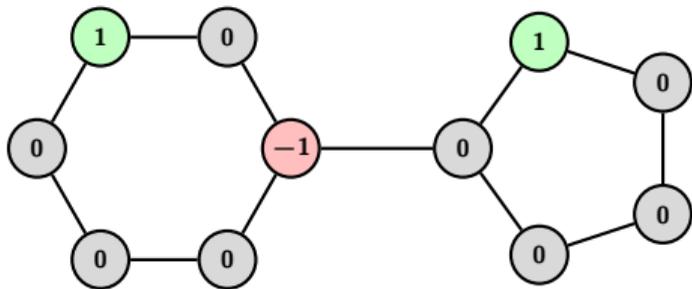
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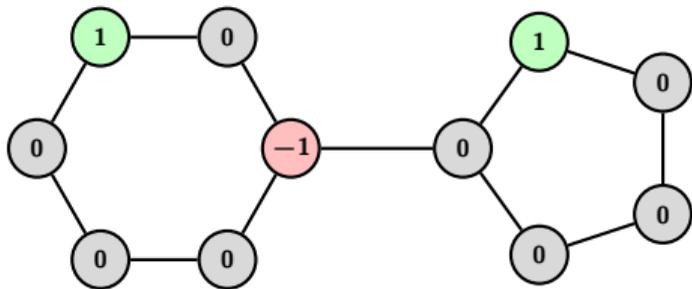
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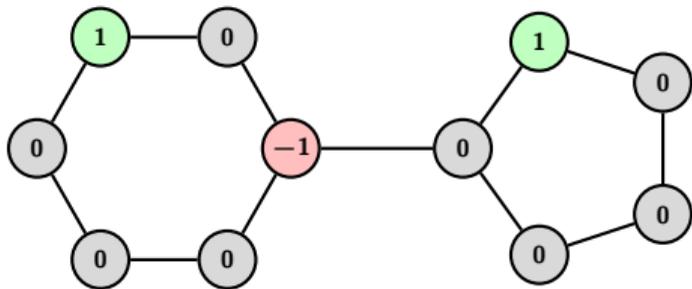
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Vague intuition: you need “as many chips as cycles” to guarantee winnability.

The genus

Classic graph theory fact: A connected graph G on V vertices has $E \geq V - 1$ edges. Equality $E = V - 1$ holds if and only if G has no cycles (i.e. is a tree).

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Definition

The *genus* of a connected graph G is

$$\text{genus}(G) = E - V + 1,$$

where E is the number of edges and V is the number of vertices.

Informally: $\text{genus}(G)$ is the number of (independent) cycles in G .

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Examples:

- If G is a tree, then $E = V - 1$, so $\text{genus}(G) = 0$.
- If $G = C_n$ is a cycle, then $E = V$, so $\text{genus}(G) = 1$.
- The barbell graph has two independent cycles, so $\text{genus}(G) = 2$.

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Theorem

If G has genus g , then

$$\deg D \geq g \implies D \text{ is winnable.}$$

This is a consequence of the *Riemann–Roch theorem*, which I'll tell you about soon.

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But I'll start saying “divisor” now, because it's in our code. Just remember:

Degree d divisor \longleftrightarrow configuration of d forints/chips

Interlude: Lean 4

Lean 4 is a programming language and proof assistant: it lets us write definitions precisely, state theorems formally, and have the computer verify proofs line by line.

In collaboration with Dhyey Mavani (Amherst '25), we recently formalized core results on chip-firing games on finite graphs in Lean 4.



Dhyey Mavani

Graphs, divisors, and degree:

```
structure CFGraph (V : Type) [DecidableEq V] [Fintype V] [Nonempty V] where
  (edges : Multiset (V × V))
  (loopless : isLoopless edges)
```

```
def isLoopless (edges : Multiset (V × V)) : Prop :=
  ∀ v, (v, v) ∉ edges
```

```
def CFDiv (V : Type) := V → ℤ -- ‘‘CFDiv’’ means ‘‘Chip Configuration’’
```

```
def genus (G : CFGraph V) : ℤ :=
  Multiset.card G.edges - Fintype.card V + 1
```

```
def deg : CFDiv V → ℤ :=
  λ D => ∑ v, D v
```

Connectedness and winnability:

```
def graph_connected (G : CFGraph V) : Prop :=  
  ∀ S : Finset V, (∃ (v w : V), v ∈ S ∧ w ∉ S) →  
    (∃ v ∈ S, ∃ w ∉ S, num_edges G v w > 0)
```

```
def winnable (G : CFGraph V)  
  (D : CFDiv V) : Prop :=  
  ∃ D' : CFDiv V,  
    (∀ v, D' v ≥ 0)  
    ∧ linear_equiv G D D'
```

Note: “linearly equivalent” is the general term of “reachable by chip-firing moves.”

The definitions match the mathematics closely, but every hidden convention has to be made explicit.

Interlude: A Lean Theorem

One key theorem, in Lean style:

```
lemma winnable_of_deg_ge_genus {G : CFGraph V} (h_conn : graph_connected G) (D : CFDiv V) :  
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A taste of Lean4 philosophy

This “theorem” is actually a *function*! Its input is a graph, a divisor, a *proof* that the graph is connected, and a *proof* that $d \geq g$, and its output is a proof that D is winnable.

In this case that function is constructive: there’s an algorithm to find the chip-firing moves.

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Thanks to the magic of Github, this omission is recorded for posterity in [Commit 87d9193](#), which remembers the 52 lines of code we needed to change. For example:

```
374 359 theorem maximal_unwinnable_deg
375 - (G : CFGraph V) (D : CFDiv V) :
360 + {G : CFGraph V} (h_conn : graph_connected G) (D : CFDiv V) :
376 361 maximal_unwinnable G D → deg D = genus G - 1 := by
377 362 intro h_max_unwin
378 363
379 364 let q := Classical.arbitrary V
380 365
381 - have h_equiv_max_unwin := maximal_unwinnable_char G q D
366 + have h_equiv_max_unwin := maximal_unwinnable_char h_conn q D
382 367 rcases h_equiv_max_unwin.mp h_max_unwin with (c, h_c_max_super, D', h_D'_qred,
h_equiv_D_D', h_D'_eq)
```

The gonality game

Back to math! Fix a graph G and an integer $k \geq 1$.

- 1 Alice chooses any divisor D of degree k on G .
- 2 Bob chooses a vertex and removes one chip there.
- 3 Alice tries to repair the resulting debt using chip-firing.

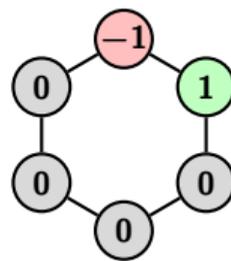
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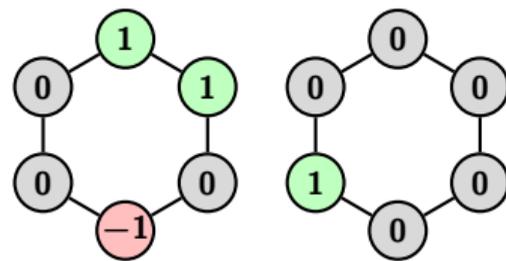
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$k = 1$

Bob wins



$k = 2$

Alice wins

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The gonality of a graph

Definition

The *gonality* of a graph G is the minimum integer k such that Alice has a winning strategy in the gonality game on G .

In other words: there exists D of degree k such that for every vertex v , $D - v$ is winnable.

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The gonality is extremely mysterious! Much remains unknown. The most prominent question is to prove or disprove:

Gonality conjecture (Baker '07)

For every connected graph G of genus g , $\text{gonality}(G) \leq \lceil \frac{1}{2}g + 1 \rceil$.

What do we know about gonality?

- **Many specific families' gonalitys have been computed or bounded:** For example, Ralph Morrison's **Williams/SMALL REU groups** have proved some *bounds* on gonality for glued grid graphs, queen's graphs, chess graphs, circulant graphs, and others, including some exact computations.

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- I highly recommend this paper as a friendly introduction to chip-firing in general, and gonality in particular:

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Chip-firing on the Platonic solids: a primer for studying graph gonality

by Marchelle Beougher, Kexin Ding, Max Everett, Robin Huang, Chan Lee, Ralph Morrison, and Ben Weber.

<https://arxiv.org/abs/2407.05158>

A gonality challenge for Lean4!

One of the few general facts about graph gonality is the following theorem of Baker.

Theorem (Baker '07)

For any connected graph G of genus g , there exists some $n \in \mathbb{N}$ such that, if we subdivide each edge of G into n edges, the resulting graph G_n has

$$\text{gonality}(G_n) \leq \left\lceil \frac{1}{2}g + 1 \right\rceil.$$

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A gonality challenge for Lean4!

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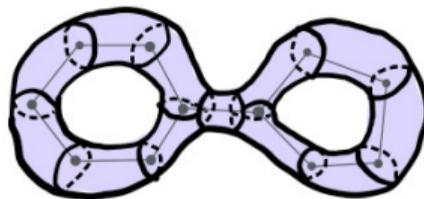
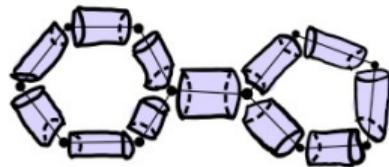
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Challenge

Give a purely combinatorial proof of this theorem, formalized in Lean4.

Algebraic geometry?!

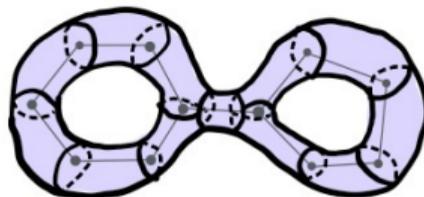
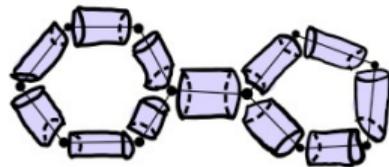
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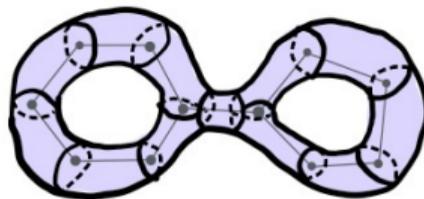
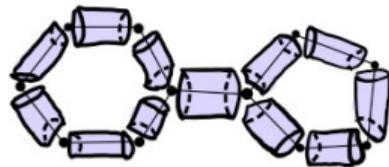


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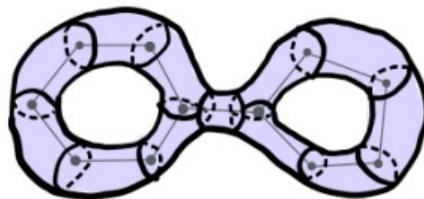
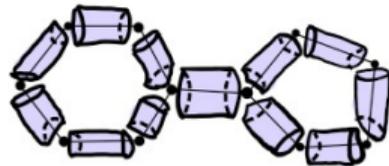
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A jewel in the crown of this analogy is the *Riemann–Roch theorem for graphs*, which I will tell you about now...



The rank game

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Think: $r(D) =$ “how winnable” D is.

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So all three of our games can be understood in terms of this one notion: the rank $r(D)$. Therefore this definition is a centerpiece of our Lean4 project.

The Riemann–Roch theorem for graphs

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Theorem (Riemann–Roch for graphs; Baker–Norine '07)

For all chip configurations D on G ,

$$r(D) = \underbrace{\deg D - g}_{\text{main term}} + \underbrace{r(K_G - D) + 1}_{\text{correction term } (\geq 0)}$$



Bernhard Riemann
1826–1866



Gustav Roch
1839–1866



Matt Baker



Sergey Norine

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- 1 Each of the g cycles is one *obstruction* to winning the rank game.
- 2 So the *expected* rank is $\deg D - g$.
- 3 *Special configurations* can have larger rank. Alice should hope for those in the rank game!

The canonical configuration

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This configuration is important because it is a “center of reflection” for a certain symmetry: the “maximal unwinnable configurations” are symmetric around it. But that's a story for another day (or you can read about it in our Lean4 docs!).

The formalization

Here's a peek at what this looks like in Lean4 syntax.

```
theorem riemann_roch_for_graphs {G : CFGraph V} (h_conn : graph_connected G) (D : CFDiv V)
  :
  rank G D - rank G (canonical_divisor G - D) = deg D - genus G + 1
:= by

let K := canonical_divisor G

-- Get key inequality
have h_ineq := rank_degree_inequality h_conn D

-- Get reverse inequality by applying to K-D
have h_ineq_rev := rank_degree_inequality h_conn (K-D)

-- Get degree of canonical divisor
have h_deg_K : deg (canonical_divisor G) = 2 * genus G - 2 :=
  degree_of_canonical_divisor G
```

The formalization

Here's the end of the same proof.

```
-- Since rank is an integer and we have bounds, equality must hold
suffices rank G D - rank G (K-D) ≥ deg D - genus G + 1 ∧
          rank G D - rank G (K-D) ≤ deg D - genus G + 1 from
le_antisymm (this.2) (this.1)
```

constructor

```
· -- Lower bound
  linarith
· -- Upper bound
  rw [deg.map_sub, h_deg_K] at h_ineq_rev
  have : canonical_divisor G - (K-D) = D := by
    rw [sub_sub_self]
  rw [this] at h_ineq_rev
  linarith
```

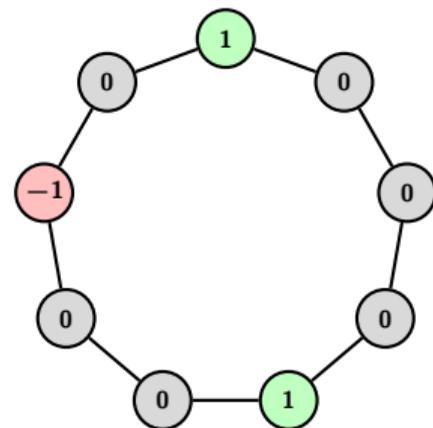
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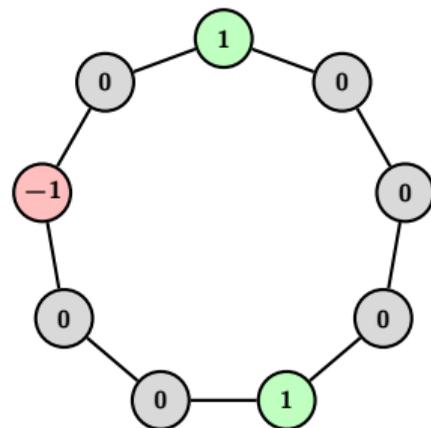
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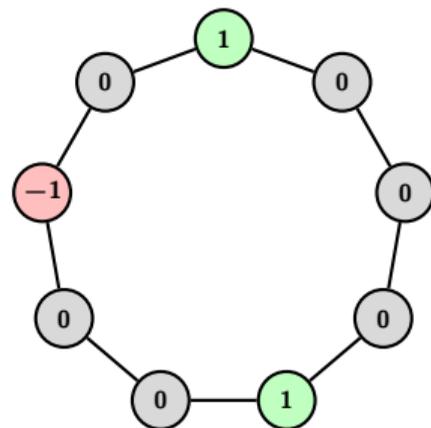
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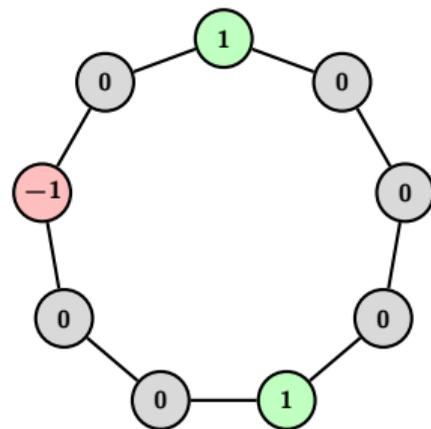
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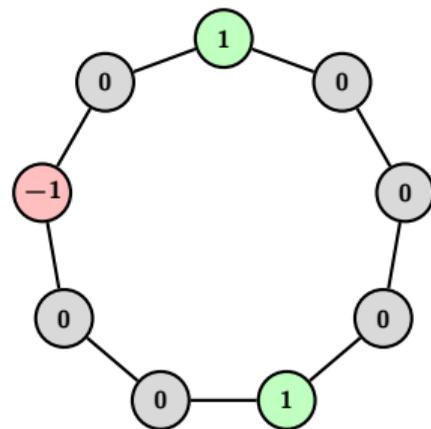
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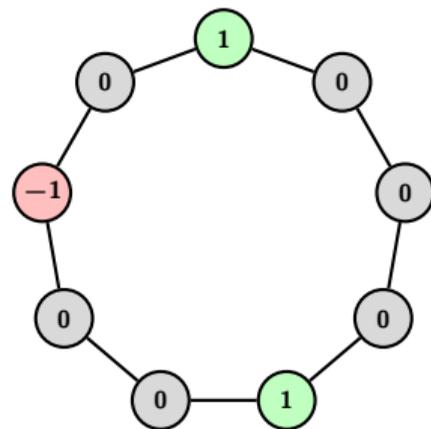
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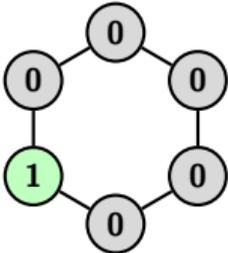
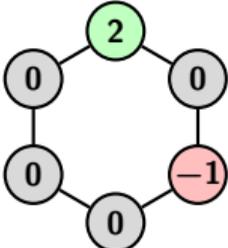
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So D is winnable! No fussing with “balancers” needed; just One Big Theorem.



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Happy chip-firing!



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