TRANSMISSION PERMUTATIONS AND DEMAZURE PRODUCTS IN HURWITZ–BRILL–NOETHER THEORY

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Abstract. We study a stratification, indexed by permutations, of the Picard scheme of a curve with two marked points, and demonstrate its utility by giving short proofs of dimension bounds from classical Brill–Noether theory and the recent subject of Hurwitz–Brill–Noether theory. A line bundle on a curve with two marked points can be special in many ways, as measured by the global sections of all of its twists by these points. All of this information is conveniently packaged into a permutation, which we call the transmission permutation. We prove that when twice-marked curves are chained together, these permutations are composed via the Demazure product; in reverse, bundles with given permutation can be enumerated via reduced decompositions of a permutation. The dimension bounds of Hurwitz–Brill–Noether theory are obtained by counting inversions in extended affine symmetric groups.

1. Introduction

How do you measure how special a line bundle on a curve is? The classical answer is of course the number that was originally called the index of speciality, $h^1(C,\mathcal{L})$; or one might just as well say $h^0(C,\mathcal{L})$, since it carries the same information. Brill–Noether theory bids you take the two numbers together, since their product is the expected codimension in $\mathrm{Pic}(C)$ of equally special line bundles. Denoting the (projective) dimension of the complete linear series of $\mathcal{L}$ by r, this product is $(r+1)(g-d+r)$.

If the curve has a marked point p, finer distinctions are possible. You may ask for the vanishing orders of $\mathcal{L}$, which measure inflection; this amounts to asking not just for $h^0(C,\mathcal{L})$ but a function $f(n) = h^0(C,\mathcal{L}(-np))$. To be fully informed, one may as well allow negative n. This information is neatly packaged in a combinatorial datum, which is named the *Weierstrass partition* in [Pfl17b]. A slick way to form it is to plot all the points $\left\{ \left(h^0\left(C,\mathcal{L}\left(-np\right)\right), h^1\left(C,\mathcal{L}\left(-np\right)\right)\right) : n \in \mathbb{Z} \right\}$ in $\mathbb{N}^2$; they form a "staircase path" tracing out a Young diagram, and *voilà*, a partition. This isn't just a gimic; the combinatorics of the partition knows about interesting geometry; its size generalizes the number $(r+1)(g-d+r)$ above and tells the expected codimension of equally special bundles, and the number of set-valued Young tableaux of content $\{1,\cdots,g\}$ tells the algebraic Euler characteristic of the Brill–Noether variety [CP21, ACT22].

But why stop at one marked point? How should one measure how special a line bundle on a twice-marked curve (C,p,q) is? One answer is to track two functions $h^0(C,\mathcal{L}(-np))$ and $h^0(C,\mathcal{L}(-nq))$. This has been the standard approach going back to the generalized Brill–Noether theorem of [EH86], and generalizes nicely to three or more marked points. The story is particularly nice for two marked points, which is the most for which the story behaves well in positive characteristic. In place of a partition, one can still build a combinatorial datum, a *skew tableau*. The size of the skew tableaux tells expected codimension, set-valued tableaux tell the Euler characteristic, and the corners of the tableaux inform you of the singular locus for general (C,p,q) [COP19, CP21, ACT22, TiB21].

This paper, along with [Pfl25], aims to promote a different, richer combinatorial datum for twice-marked curves. This datum remembers more: a two-variable function $f(a,b) = h^0(C,\mathcal{L}(ap-bq))$

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that knows not just how p, q are inflected, but how they interact. This datum is a permutation $\mathbb{Z} \to \mathbb{Z}$, which we dub herein the *transmission permutation.* Like the Weierstrass partition, this permutation is not a mere bookkeeping device: its combinatorics knows interesting geometric information. In place of the number $(r+1)(g-d+r)$ or the size of a (skew) tableaux, the *number of inversions* of the permutation predicts the codimension of equally special divisors, and reduced words take the place of tableaux in enumerative questions. We also get a bonus: by consider permutations in the extended affine symmetric group, we learn not about general points in $\mathcal{M}_{g,2}$, but general points in a Hurwitz space (as explained below). Intimately linked to this story is a curious associative operation on permutations related to tropical matrix multiplication called the Demazure product.

The aim of this paper is modest: to demonstrate the utility of this construction with a short unified proof of the existence of Brill–Noether and Hurwitz–Brill–Noether general curves. See also the partner paper [Pfl21], which develops similar notions in the tropical context.

Remark 1.1. The permutations we use in this paper differ from those used in [Pfl25], in that increasing, rather than decreasing permutations are the most generic; this is because we describe $h^0(C, \mathcal{L}(ap-bq))$, rather than $h^0(C, \mathcal{L}(-ap-bq))$, via permutations. The choice in this paper allows a cleaner use of the Demazure product, and reflects the fact that, when chaining curves together, a generic degree g line bundle has no effect, and thus corresponds to multiplying by the identity permutation.

1.1. **Transmission loci.** Let (C, p, q) be a twice-marked smooth curve. If $\mathcal{L}$ is a line bundle on C, there is a unique permutation $\tau = \tau^{p,q}_{\mathcal{L}} : \mathbb{Z} \to \mathbb{Z}$ characterized by

$$h^0(C, \mathcal{L}(ap-bq)) = \#\{n \geq b : \tau(n) \leq a\}, \text{ and} \tag{1}$$

$$h^1(C, \mathcal{L}(ap-bq)) = \#\{n < b : \tau(n) > a\}, \text{ for all } a, b \in \mathbb{Z}. \tag{2}$$

For example, if $\tau = \iota_{d-g}$, where $\iota_{d-g}(n) = d-g+n$ for all $n \in \mathbb{Z}$, then Equations (1) and (2) say that every twist $\mathcal{L}(ap-bq)$ is nonspecial (either $h^0 = 0$ or $h^1 = 0$). So increasing permutations correspond to the most generic situation, and inversions in τ signal special divisors.

Note in particular that

$$\#\{(m,n) : m < 0 \leq n, \tau(n) \leq 0 < m\} = h^0(C, \mathcal{L}) h^1(C, \mathcal{L}) = (r+1)(g-d+r)$$

if $h^0(C, \mathcal{L}) = r+1$, so the expected codimension from classical Brill–Noether theory is a lower bound on the number of inversions of τ. This permutation is *almost-sign-preserving,* by which we mean that it changes the sign of finitely many integers. We denote the group of such permutations by ASP. The existence of $\tau^{p,q}_{\mathcal{L}}$ is explained in Section 3. This permutation conveniently packages substantial geometric information about $\mathcal{L}$ beyond vanishing orders, including the presence of nodes, bitangents and other features linking p to q (see [Pfl25, Figure 1]). The degree of $\mathcal{L}$ is encoded by a number that we call the *shift* of the permutation, and defined by

$$\chi_\alpha = \#\{n \geq 0 : \alpha(n) < 0\} - \#\{n < 0 : \alpha(n) \geq 0\}.$$

The definition of $\tau^{p,q}_{\mathcal{L}}$ shows that $\chi_{\tau^{p,q}_{\mathcal{L}}} = \chi(C, \mathcal{L}(-p)) = d-g$, where d is the degree of $\mathcal{L}$ and g is the genus of C.

In the other direction, any $\tau \in$ ASP defines a subvariety of $\mathrm{Pic}^{\chi_\alpha + g}(C)$. Define

$$\begin{aligned} W^\tau(C,p,q) &= \Big\{[\mathcal{L}] \in \mathrm{Pic}^{\chi_\tau+g}(C) : \ h^0(C, \mathcal{L}(ap-bq)) \geq \# \ \{n \geq b : \tau(n) \leq a\} \text{ for all } a,b \in \mathbb{Z}\Big\}, \\ &= \Big\{[\mathcal{L}] \in \mathrm{Pic}^{\chi_\tau+g}(C) : \ h^1(C, \mathcal{L}(ap-bq)) \geq \# \ \{n < b : \tau(n) > a\} \text{ for all } a,b \in \mathbb{Z}\Big\}. \end{aligned}$$

We call this the *τ-transmission locus.* The fact that these two equations are equivalent comes from Riemann-Roch and Equation (6) below. The formulation via a bound on h^1 is better-suited for a family of curves, as explained in Remark 5.2.

We are particularly interested in *extended k-affine permutations.*

Definition 1.2. Let $k = 0$ or $k \geq 2$ be an integer. Denote by $\widetilde{\Sigma}_k$ the group of permutations $\alpha \in \mathrm{ASP}$ such that $\alpha(n+k) = \alpha(n)+k$ for all $n \in \mathbb{Z}$. In particular, $\widetilde{\Sigma}_0 = \mathrm{ASP}$ (this notation differs slightly from that of [Pfl21]). For $k \geq 2$, these are called the *extended affine symmetric groups.*

The word "extended" is present because when $k \geq 2$ the "affine symmetric group" is the subgroup of permutations such that $\sum_{n=0}^{k-1}(\alpha(n) - n) = 0$, or equivalently $\chi_\alpha = 0$.

We call a twice-marked curve (C, p, q) *k-torsion* if $kp \sim kq$ as divisors, and say that k is the *torsion order* if it generates $\{n \in \mathbb{Z} : np \sim nq\}$. The definition of $\tau^{p,q}_{\mathcal{L}}$ shows that if (C, p, q) is k-torsion, $\tau^{p,q}_{\mathcal{L}} \in \widetilde{\Sigma}_k$ for all $\mathcal{L}$. The converse is also true; if $\tau^{p,q}_{\mathcal{O}_C} \in \widetilde{\Sigma}_k$, then it follows that $h^0(C, \mathcal{O}_C) = h^0(C, \mathcal{O}_C(kp - kq)) = 1$, so $kp \sim kq$.

An *inversion* of α is a pair (m, n) such that $m < n$ and $\alpha(m) > \alpha(n)$, and two inversions $(m, n), (m', n')$ are called k-equivalent if $m' - m = n' - n \equiv 0 \pmod{k}$. The complexity of such permutations is measured by $\mathrm{inv}_k(\alpha)$, the number of k-equivalence classes of inversions; $\mathrm{inv}_0(\alpha)$ is simply the number of inversions. Note that "extended 0-affine" and "0-torsion" are vacuous conditions, but we allow them to state results in a uniform way.

Studying these transmission loci provide a route to studying the classical Brill–Noether varieties, as well as the more recently studied *splitting loci* of Hurwitz–Brill–Noether theory. In particular,

(1) Given g, r, d, there is a permuation γ^r_{d-g} such that, for C a genus g curve, the Brill–Noether locus $W^r_d(C)$ is equal to $W^{\gamma^r_{d-g}}(C, p, q)$, regardless of p, q. The number of inversions of γ^r_{d-g} is $(r + 1)(g - d + r)$, the expected codimension of $W^r_d(C)$.
(2) Given any $k \geq 2$ and splitting type $\vec{e} \in \mathbb{Z}^k$, there is a permutation $\gamma_{\vec{e}}$ such that for any (C, p, q) with $kp \sim kq$, the splitting locus $W^{\vec{e}}(C, kp)$ is equal to $W^{\gamma_{\vec{e}}}(C, p, q)$. The number of inversions of $\gamma_{\vec{e}}$ is equal to $u(\vec{e})$, the expected codimension of $W^{\vec{e}}(C, kp)$.

See Section 6 for terminology and the precise statements. The construction of these permutations is not novel; $\gamma_{\vec{e}}$ appears in slightly different form in [LLV25, Theorem 1.4], for example. The novelty in the present paper is placing them in the context of transmission loci, and offering the Demazure product as a useful device for inductive arguments.

1.2. **The Demazure product.** The fundamental tool in our argument is the *Demazure product* on ASP. This is an associative product $\star$ with several nice characterizations, whose properties are developed in detail in [Pfl22] and summarized in Section 2 here. Briefly, $\alpha \star \beta$ can be obtained by decreasing α, β in Bruhat order until a "reduced product" is obtained, and then multiplying them; it is also characterized by a type of matrix multiplication over the min-plus semiring. The fundamental observation on which this entire paper turns is that, if $(C_1, p_1, q_1), (C_2, p_2, q_2)$ are two twice-marked chains (or smooth curves), and q_1 is glued to p_2 to form (X, p_1, q_2), then for any line bundle $\mathcal{L}$ on X restricting to $\mathcal{L}_1, \mathcal{L}_2$ on C_1, C_2,

$$\tau^{p_1,q_2}_{\mathcal{L}} = \tau^{p_1,q_1}_{\mathcal{L}_1} \star \tau^{p_2,q_2}_{\mathcal{L}_2}. \tag{3}$$

This is proved in Corollary 3.6. In reverse, transmission loci $W^\tau(X, p_1, q_2)$ may be decomposed into a union of products $W^\alpha(C_1, p_1, q_1) \times W^\beta(C_2, p_2, q_2)$, where the union is taken over *reduced* products $\alpha\beta = \tau$. This is proved in Proposition 3.7. What this means for our analysis is that transmission permutations are extremely well-suited to inductive arguments in which curves are repeatedly split into two curves joined at a node. They share this feature with limit linear series, and for this reason *elliptic chains* serve as a natural endpoint for degeneration, just as they do in many applications of limit linear series. If nothing else, my hope in writing this paper is to make the case that the Demazure product is the most natural underlying combinatorial mechanism to understand why elliptic chains have been so successful in Brill–Noether theory.

1.3. **Results.** We are concerned in this paper with the following genericity condition. In light of the discussion above, this notion implies classical Brill–Noether generality ($k = 0$) and Hurwitz–Brill–Noether generality ($k \geq 2$). In both cases, "generality" refers only to loci having the expected dimension, and not to smoothness or stronger conditions.

Definition 1.3. Let $k = 0$ or $k \geq 2$, and let (C, p, q) be a twice-marked curve or chain. We say that (C, p, q) has *k-general transmission* if

(1) Every transmission permutation $\tau^{p,q}_{\mathcal{L}}$ on (C, p, q) is in $\widetilde{\Sigma}_k$; and
(2) For all $\tau \in \widetilde{\Sigma}_k$, any component of $W^\tau(C, p, q)$ has codimension at least $\mathrm{inv}_k(\tau)$.

In particular, if $\mathrm{inv}_k(\tau) > g$, including the possibility that $\mathrm{inv}_k(\tau) = \infty$, then $W^\tau(C, p, q)$ is empty.

We conjecture (Conjecture 7.1) that in fact on *any* (C, p, q) with $kp \sim kq$, every component of $W^\tau(C, p, q)$ has codimension *at most* $\mathrm{inv}_k(\alpha)$. The is true for the permutations γ^r_{d-g} and $\gamma_{\vec{e}}$ mentioned above by degeneracy locus arguments [Kem71, KL72, KL74] and Larson's theory of splitting loci [Lar21b, Lar21a], respectively. If correct, this definition could be changed to say that $W^\tau(C, p, q)$ is equidimensional of codimension $\mathrm{inv}_k(\tau)$ when $\mathrm{inv}_k(\tau) \leq g$. See Section 7 for some discussion and further conjectures.

Our main result is the following. Here, $\mathcal{M}_{g,2}$ denotes the moduli stack of twice-marked smooth curves, and $\mathcal{H}_{g,k,2}$ denotes the substack of (C, p, q) such that $kp \sim kq$, or alternatively the space of degree-k covers $\pi : C \to \mathbb{P}^1$ with two marked points of total ramification.

Theorem 1.4. *A very general twice-marked curve (C, p, q) in $\mathcal{M}_{g,2}$ has 0-general transmission, and such curves are Brill–Noether general; for all $k \geq 2$ a very general point in some component of $\mathcal{H}_{g,k,2}$ has k-general transmission, and such curves are Hurwitz–Brill–Noether general.*

The phrase "some component of" above is necessary becuase, in characteristic p, $\mathcal{H}_{g,k,2}$ may be reducible. In characteristic 0 this phrase is not needed. The $k = 0$ case of this theorem follows from [Pfl25, Theorem 1.12], which also gives an existence and smoothness statement, and uses somewhat different methods. I do not know if the "versality of flags" point of view in that paper can be adapted to the $k \geq 2$ situation.

We will consider certain degenerations, namely to chains of twice-marked curves. For our purposes, a *twice-marked chain* is a twice-marked nodal curve (X, p_1, q_ℓ) obtained from ℓ twice-marked smooth curves (C_i, p_i, q_i), $1 \leq i \leq \ell$, by gluing q_i to p_{i+1} for $1 \leq i < \ell$. In particular, whenever we say "twice-marked chain," we will always assume that the marked points are at opposite ends of the chain. We allow $\ell = 1$, so that "twice-marked chains" include "twice-marked smooth curves." We will extend the definition of transmission permutations from smooth curves to such chains in Section 3. With this terminology in place, we will prove:

(1) In a family of twice-marked chains (possibly including smooth curves), the function $(C, p, q) \mapsto \dim W^\tau(C, p, q)$ is upper semicontinuous (Theorem 5.1).
(2) A genus-1 curve has k-general transmission if and only if the marked points differ by torsion of order exactly k (Theorem 4.1).
(3) General transmission is "chainable:" A twice-marked chain has k-general transmission if and only if every component in the chain has k-general transmission (Theorem 3.9).

In particular, these results together imply Theorem 1.4, by considering elliptic chains with torsion order k on all components. We must say "very general" in that theorem since countably many permutations τ must be considered.

Conventions

We work over an algebraically closed field $\mathbb{F}$ of any characteristic. By *curve* we always mean a connected proper nodal curve. A *twice-marked curve* is a curve with two distinct marked smooth points, and a *twice-marked chain* is always assumed to have its marked points at opposite ends of the chain.

The symbol $\mathbb{N}$ denote *nonnegative* numbers. The symbol δ always denotes an indicator function, equal to 1 if the statement within is true, and 0 otherwise. A *permutation* always refers to a permutation of $\mathbb{Z}$.

2. The Demazure product on ASP

This section summarizes material about the Demazure product on ASP. This content is explained in detail in [Pfl22], and much of it is standard material for Coxeter groups.

The Demazure product on ASP may be defined via the functions

$$s_\alpha(a,b) = \#\{n \geq b : \alpha(n) < a\}$$

associated to permutations $\alpha \in \mathrm{ASP}$. These are called *(submodular) slipface functions* in [Pfl22], and are closely related to *rank functions* on finite symmetric groups. The Demazure product is uniquely characterized by the "min-plus matrix multiplication" equation

$$s_{\alpha\star\beta}(a,b) = \min_{\ell\in\mathbb{Z}} \{s_\alpha(a,\ell) + s_\beta(\ell,b)\}\,. \tag{4}$$

This equation undoubtedly appears strange at first; I suggest the reader "try it out" by checking the special case: if α, β^{-1} have no inversions in common, then $\alpha \star \beta = \alpha\beta$ (the converse also holds); this is easiest to understand when β is a simple transposition.

We will require a criterion for obtaining a permutation from a slipface function.

Corollary 2.1 ([Pfl22, Proposition 7.12])**.** *Let $s : \mathbb{Z}^2 \to \mathbb{N}$ be a function. Suppose that*

(1) There exists integers M, χ such that $a - b \leq -M$ implies $s(a,b) = 0$ and $a - b \geq M$ implies $s(a,b) = \chi + a - b$; and
(2) For all $a, b \in \mathbb{Z}$, $s(a+1,b) - s(a,b) - s(a+1,b+1) + s(a,b+1) \geq 0$ (s is submodular*).*

Then there exists a unique permutation $\alpha \in \mathrm{ASP}$ such that $s(a,b) = s_\alpha(a,b)$. The shift of α is the the number χ mentioned in criterion (1). This permutation has bounded difference*, meaning that $|\alpha(n) - n|$ is bounded for $n \in \mathbb{Z}$.*

The functions s_α also define the *Bruhat order* on ASP: $\alpha \leq \beta$ means $s_\alpha(a,b) \leq s_\beta(a,b)$ for all $a, b \in \mathbb{Z}$. We will almost never use Bruhat order to compare permutations with different shifts. Bruhat order provides a second, perhaps more intuitive definition of the Demazure product: it is the Bruhat-maximum of all ordinary products of Bruhat-smaller permutations:

$$\alpha \star \beta = \max\{\alpha_1\beta_1 : \alpha_1 \leq \alpha, \beta_1 \leq \beta\}. \tag{5}$$

The shift of a permutation $\alpha \in \mathrm{ASP}$ determines the asymptotic behavior of s_α. This is revealed by the following identity.

$$s_\alpha(a,b) - s_{\alpha^{-1}}(b,a) = \chi_\alpha + a - b \tag{6}$$

In particular, if $\chi_\alpha = \chi_\beta$ then $\alpha \leq \beta$ if and only if $\alpha^{-1} \leq \beta^{-1}$.

The shift map $\alpha \mapsto \chi_\alpha$ is a homomorphism for both $\star$ and ordinary multiplication.

$$\chi_{\alpha\star\beta} = \chi_{\alpha\beta} = \chi_\alpha + \chi_\beta \tag{7}$$

A crucial step in our argument will consist of reducing Demazure products to ordinary products. This step is provided by the following "reduction theorem" from [Pfl22].

Definition 2.2. A tuple $(\alpha_1, \cdots, \alpha_\ell)$ is called *reduced* if $\mathrm{Inv}(\alpha_1 \cdots \alpha_\ell)$ is equal to the *disjoint* union of the sets $\Big\{ \Big((\alpha_{n+1} \cdots \alpha_\ell)^{-1}(u), (\alpha_{n+1} \cdots \alpha_\ell)^{-1}(v) \Big) : (u,v) \in \mathrm{Inv}(\alpha_n) \Big\}$ for $1 \leq n \leq \ell$.

Theorem 2.3. *Let $\alpha_1, \cdots, \alpha_\ell, \gamma \in \widetilde{\Sigma}_k$, and suppose $\alpha_1 \star \cdots \star \alpha_\ell \geq \gamma$ and $\sum \chi_{\alpha_n} = \chi_\gamma$. Then there exists a* reduced *ℓ-tuple $(\beta_1, \cdots, \beta_\ell)$ in $\widetilde{\Sigma}_k$ such that $\chi_{\beta_i} = \chi_{\alpha_i}$ and $\beta_i \leq \alpha_i$ for all i, and $\beta_1 \star \cdots \star \beta_\ell = \beta_1 \cdots \beta_\ell = \gamma$. In particular, $\sum_{i=1}^{\ell} \mathrm{inv}_k(\beta_i) = \mathrm{inv}_k(\gamma)$.*

3. Transmission permutations on smooth curves and chains

Let (C,p,q) be a twice-marked smooth curve of genus g, and $\mathcal{L}$ be a degree d line bundle on C. Define the *transmission function* $s^{p,q}_{\mathcal{L}}$ by

$$s^{p,q}_{\mathcal{L}}(a,b) = h^0\Big(C, \mathcal{L}\big(\,(a-1)\,p - bq\big)\Big).$$

We write $a-1$ rather than a in this definition because it is a necessary (though headache-inducing!) correction to ensure that transmission permutations are combined using the Demazure product. Let $\chi = d-g$. Riemann-Roch implies that $s^{p,q}_{\mathcal{L}}$ satisfies Criterion (1) of Corollary 2.1, and Criterion (2) follows from the observation that, letting $V_{a,b} = H^0\Big(C, \mathcal{L}\big((a-1)p - bq\big)\Big)$ and regarding all these as subspaces of $H^0(C \backslash \{p,q\}, \mathcal{L})$,

$$s^{p,q}_{\mathcal{L}}(a+1,b) - s^{p,q}_{\mathcal{L}}(a,b) - s^{p,q}_{\mathcal{L}}(a+1,b+1) + s^{p,q}_{\mathcal{L}}(a,b) = \dim V_{a+1,b} / \left(V_{a,b} + V_{a+1,b+1}\right) \geq 0.$$

Furthermore, Equation (6) and Riemann-Roch imply $s_{\tau^{-1}}(b,a) = h^1\Big(C, \mathcal{L}\big((a-1)p - bq\big)\Big)$, hence

Lemma 3.1. *For any twice-marked smooth curve (C,p,q) and line bundle $\mathcal{L}$, there exists a permutation $\tau = \tau^{p,q}_{\mathcal{L}}$ satisfying $s^{p,q}_{\mathcal{L}} = s_\tau$ and $s^{q,p}_{\omega_C(p+q) \otimes \mathcal{L}^\vee} = s_{\tau^{-1}}$, and therefore Equations (1), (2).*

Remark 3.2. If ι_n denotes the shift permutation $\iota_n(m) = m - n$, then $\chi_{\iota_n} = n$, and

$$s_{\iota_n \alpha}(a,b) = s_\alpha(a+n,b)$$

for all $a,b \in \mathbb{Z}$, so $\iota_n \tau^{p,q}_{\mathcal{L}} = \tau^{p,q}_{\mathcal{L}(np)}$ for all line bundles $\mathcal{L}$, and $W^\tau(C,p,q) \cong W^{\iota_n \tau}(C,p,q)$ for all permutations τ, via the "twist by np" map. This is a convenient way to reduce certain statements to the shift 0, or alternatively to the degree 0, case.

3.1. **Chains of twice-marked curves.** We now extend our definition of transmission loci to certain nodal curves, namely twice-marked chains. We work with chains for simplicity, but similar definitions can be made for curves of compact type, and the reader familiar with the theory of *enriched structures* will see that these definitions naturally extend to that context as well. The primary catch is that it may not be the case that all transmission functions are submodular, so transmission permutations may not exist, in these more general settings. This is related to the fact that in the tropical context [Pfl21], transmission functions are not submodular in general.

Fix a single twice-marked chain (X, p_1, q_ℓ) obtained by joining $(C_1, p_1, q_1), \cdots, (C_\ell, p_\ell, q_\ell)$. For each node $q_i \in \{q_1, \cdots q_{\ell-1}\}$, there is a line bundle (unique up to isomorphism) $\mathcal{Y}_i$ such that

$$\mathcal{Y}_i\mid_{C_i} \cong \mathcal{O}_{C_i}(-q_i),\ \mathcal{Y}_i\mid_{C_{i+1}} \cong \mathcal{O}_{C_{i+1}}(p_{i+1}) \text{ and } \mathcal{Y}_i\mid_{C_j} \cong \mathcal{O}_{C_j} \text{ for all } j \neq i, i+1.$$

For any $\vec{n} \in \mathbb{Z}^{\ell-1}$, define $\mathcal{Y}(\vec{n}) = \bigotimes_{i=1}^{\ell-1} \mathcal{Y}_i^{\otimes n_i}$. Denote by $\mathrm{Pic}^{d,p_1}(X)$ is the component of the Picard scheme parameterizing line bundles of degree d on C_1 and 0 on $C_2, \cdots, C_\ell$.

Definition 3.3. The *transmission function* $s^{p_1,q_\ell}_{\mathcal{L}} : \mathbb{Z}^2 \to \mathbb{N}$ of a line bundle $\mathcal{L}$ on X is

$$s^{p_1,q_\ell}_{\mathcal{L}}(a,b) = \min_{\vec{n} \in \mathbb{Z}^{\ell-1}} h^0(X, \mathcal{L}((a-1)p - bq) \otimes \mathcal{Y}(\vec{n})).$$

For any $\tau \in \mathrm{ASP}$, let $d = \chi_\tau + g$ and define the *transmission locus* $W^\tau(X, p_1, q_\ell)$ to be

$$W^\tau(X,p_1,q_\ell) = \Big\{[\mathcal{L}] \in \mathrm{Pic}^{d,p_1}(X) : \ s_{\mathcal{L}}^{p_1,q_\ell}(a,b) \geq s_\tau(a,b) \text{ for all } a,b \in \mathbb{Z}\Big\}.$$

Note by Equation (6) and Riemann-Roch, this bound on $h^0(X, \mathcal{L}(ap - bq) \otimes \mathcal{Y}(\vec{n}))$ is equivalent to bounding $h^1(X, \mathcal{L}(ap - bq) \otimes \mathcal{Y}(\vec{n}))$ by $s_{\tau^{-1}}(b, a+1)$.

To carry out our analysis, we require an alternative form for $s_{\mathcal{L}}^{p_1,q_\ell}$ stated purely in terms of the individual line bundles. To obtain it, we require a basic lemma about line bundles on nodal curves. The reader who is frustrated with the insidious "-1"s that creep into many of our definition (such as that of $s_{\mathcal{L}}^{p_1,q_\ell}$ above) may direct their frustration at this lemma, where these goblins originate.

Lemma 3.4. *Let X be a nodal curve, with a node p that separates it into two nodal curves X_1, X_2. Let $\mathcal{L}$ be a line bundle on X, with $\mathcal{L}_1, \mathcal{L}_2$ the restrictions to X_1, X_2. For $n \in \mathbb{Z}$, let $\mathcal{Y}(n)$ be a line bundle on X with $\mathcal{Y}(n)\mid_{X_1} \cong \mathcal{O}_{X_1}(-np)$ and $\mathcal{Y}(n)\mid_{X_2} \cong \mathcal{O}_{X_2}(np)$. Then*

$$\min_{n\in\mathbb{Z}} \left\{h^0(X, \mathcal{L} \otimes \mathcal{Y}(n))\right\} = \min_{n\in\mathbb{Z}} \left\{h^0(X_1, \mathcal{L}_1(-np)) + h^0(X_2, \mathcal{L}_2((n-1)p))\right\}.$$

Proof. This follows from the claim: for any line bundle $\mathcal{L}$ on X,

$$h^0(X,\mathcal{L}) = \min\Big\{h^0(X_1,\mathcal{L}_1) + h^0(X_2,\mathcal{L}_2(-p)), h^0(X_1,\mathcal{L}_1(-p)) + h^0(X_2,\mathcal{L}_2)\Big\}.$$

To prove this claim, consider the exact sequence $0 \to \mathcal{L} \to \mathcal{L}_1 \oplus \mathcal{L}_2 \to \mathcal{L}\mid_p \to 0$ of sheaves on X. The last term is of course isomorphic to $\mathcal{O}_p$. Taking global sections, it follows that

$$h^0(X,\mathcal{L}) = h^0(X_1,\mathcal{L}_1) + h^0(X_2,\mathcal{L}_2) - \delta,$$

where δ is 0 if both $\mathcal{L}_1, \mathcal{L}_2$ have a base point at p, and 1 otherwise. In other words, $\delta = \max\{h^0(X_1,\mathcal{L}_1) - h^0(X_1,\mathcal{L}_1(-p)), h^0(X_2,\mathcal{L}_2) - h^0(X_2,\mathcal{L}_2(-p))\Big\}$; the claim and the lemma follow. ☐

By induction on ℓ, we can reduce $s_{\mathcal{L}}^{p_1,q_\ell}$ to transmission functions of the components as follows.

Corollary 3.5. *Let $\mathcal{L}$ be a line bundle on the chain (X, p_1, q_ℓ), and denote by $\mathcal{L}_i$ the restriction to C_i. For all $n_0, n_\ell \in \mathbb{Z}$,*

$$s_{\mathcal{L}}^{p_1,q_\ell}(n_0,n_\ell) = \min_{n_1,\cdots,n_{\ell-1}\in\mathbb{Z}} \left\{\sum_{i=1}^{\ell} s_{\mathcal{L}_i}^{p_i,q_i}(n_{i-1},n_i)\right\}.$$

In light of Lemma 3.1 and the existence and definition of the Demazure product (in this case, an iterated Demazure product), this shows that chains don't just have transmission *functions*; they too have transmission *permutations*, and attachment at a node corresponds to the Demazure product.

Corollary 3.6. *For any degree-d line bundle $\mathcal{L}$ on the chain (X, p_1, q_ℓ), there exists a permutation $\tau = \tau_{\mathcal{L}}^{p_1,q_\ell}$ of shift $\chi_\tau = d - g$ such that $s_{\mathcal{L}}^{p_1,q_\ell} = s_\tau$. This permutation is given by*

$$\tau_{\mathcal{L}}^{p_1,q_\ell} = \tau_{\mathcal{L}_1}^{p_1,q_1} \star \tau_{\mathcal{L}_2}^{p_2,q_2} \star \cdots \star \tau_{\mathcal{L}_\ell}^{p_\ell,q_\ell}.$$

Proposition 3.7. *Let $\tau \in \widetilde{\Sigma}_k$, let (X, p_1, q_ℓ) be a twice-marked chain as above, and let $d = \chi_\tau + g$, where g is the genus of X. Let W be the set of* reduced *tuples $(\alpha_1, \cdots, \alpha_\ell)$ in $\widetilde{\Sigma}_k$ that satisfy $\chi_{\alpha_1} = \chi_\tau$ and $\chi_{\alpha_i} = 0$ for $2 \leq i \leq \ell$, and such that $\alpha_1\alpha_2\cdots\alpha_\ell = \tau$. Identify* $\mathrm{Pic}^{d,p_1}(X)$ *with* $\mathrm{Pic}^d(C_1) \times \mathrm{Pic}^0(C_2) \times \cdots \times \mathrm{Pic}^0(C_\ell)$*. Then*

$$W^\tau(X,p_1,q_\ell) = \bigcup_W \prod_{i=1}^{\ell} W^{\alpha_i}(C_i,p_i,q_i).$$

Proof. A bundle $[\mathcal{L}] \in \mathrm{Pic}^{d,p_1}(X)$ lies in $W^\tau(X, p_1, q_\ell)$ if and only if $\tau^{p_1,q_1}_{\mathcal{L}_1} \star \tau^{p_2,q_2}_{\mathcal{L}_2} \star \cdots \star \tau^{p_\ell,q_\ell}_{\mathcal{L}_\ell} \geq \tau$ in Bruhat order. By assumption on $\mathcal{L}$, the shifts of these permutations are $\chi_\tau, 0, \cdots, 0$, respectively. By Theorem 2.3, this occurs if and only if there exists a reduced product $\alpha_1 \cdots \alpha_\ell = \tau$ with the same shifts and $\alpha_i \leq \tau^{p_i,q_i}_{\mathcal{L}_i}$ for all i. These inequalities are equivalent to $[\mathcal{L}] \in \prod_{i=1}^\ell W^{\alpha_i}(C_i, p_i, q_i)$. So $W^\tau(X, p_1, q_\ell)$ is equal to the union of all such products for $\alpha_1, \cdots, \alpha_\ell$ chosen from the set W. □

Example 3.8. Suppose that $\mathrm{inv}_k(\tau) = g$, and X is a chain of k-torsion twice-marked genus 1 curves. For simplicity assume $\chi_\tau = 0$ (by Remark 3.2, this does not really limit anything). It will follow from the analysis in Section 4 that every $(\alpha_1, \cdots, \alpha_\ell) \in W$ has $\mathrm{inv}_k(\alpha_n) = 1$ for all n, so in fact W consists of the set of *reduced words* for τ in the affine symmetric group, and we obtain a bijection between the points of $W^\tau(X, p_1, q_g)$ and reduced words.

Theorem 3.9. *A twice-marked chain of k-torsion curves has k-general transmission if and only if each curve in the chain has k-general transmission.*

Proof. Suppose each (C_i, p_i, q_i) has k-general transmission. Then for every choice of line bundles $\mathcal{L}_i$ on C_i, $\tau^{p_i,q_i}_{\mathcal{L}_i} \in \widetilde{\Sigma}_k$ for all i, so Corollary 3.6 implies that $\tau^{p_1,q_\ell}(\mathcal{L}) \in \widetilde{\Sigma}_k$ as well, since $\widetilde{\Sigma}_k$ is closed under $\star$. We now consider the dimension bound. For every $\tau \in \widetilde{\Sigma}_k$, every element of W has $\mathrm{inv}_k \tau = \sum_{i=1}^\ell \mathrm{inv}_k \alpha_i$, so every element of the union in that Proposition has codimension at least $\mathrm{inv}_k \tau$. It follows that (X, p_1, q_ℓ) has k-general transmission.

Conversely, suppose that the chain (X, p_1, q_ℓ) has k-general transmission. To tame an illegible nest of subscripts in our notation, define $s_n = s_{\tau^{p_n,q_n}_{\mathcal{O}_{C_n}}}$ for $n = 1, \cdots, \ell$. We claim that each twice-marked curve (C_n, p_n, q_n) has k-torsion, or equivalently that $s_n(k+1, k) \geq 1$. To see this, note that $\tau^{p_1,q_\ell}_{\mathcal{O}_X}$ is the Demazure product of all $\tau^{p_i,q_i}_{\mathcal{O}_{C_i}}$. Since the chain as a whole has k-general transmission, we have $s_1 \star \cdots \star s_\ell(k+1, k) \geq 1$. It follows that for each n,

$$\sum_{i=1}^{n-1} s_i(k+1, k+1) + s_n(k+1, k) + \sum_{i=n+1}^{\ell} s_i(k, k) \geq 1.$$

For all i, $s_i(k+1, k+1) = s_i(k, k) = 0$, since these are dimensions of spaces of sections of negative-degree line bundles. It follows that $s_n(k+1, k) = h^0(C_n, \mathcal{O}_{C_n}(kp_n - kq_n)) \geq 1$, which proves the claim. There all transmission permutations on all components (C_n, p_n, q_n) lie in $\widetilde{\Sigma}_k$.

We must now verify the codimension bound on each component. Let τ be any shift-0 permutation, and let $1 \leq n \leq \ell$. Decompose $W^\tau(X, p_1, q_\ell)$ as in Proposition 3.7 and define a tuple $(\alpha_1, \cdots, \alpha_\ell)$ by $\alpha_n = \tau$ and $\alpha_i = \iota_0$ (the identity) otherwise. This tuple is an element of set W, and $W^{\alpha_i}(C_i, p_i, q_i)$ is all of $\mathrm{Pic}^0(C_i)$ if $i \neq n$, and $W^\tau(C_n, p_n, q_n)$ when $i = n$. So $\prod_{i=1}^\ell W^{\alpha_i}(C_i, p_i, q_i)$ has the same codimension in $\mathrm{Pic}^{0,p_1}(X)$ as $W^\tau(C_n, p_n, q_n)$ does in $\mathrm{Pic}^0(C_n)$, which is therefore at least $\mathrm{inv}_k(\tau)$. By Remark 3.2, this implies that the same holds for all τ of *any* shift. So all (C_n, p_n, q_n) have k-general transmission. □

4. The genus 1 case

The story of transmission permutations is particularly simple in genus 1, which makes it an excellent base case. This section proves

Theorem 4.1. *A genus-1 twice-marked curve (E, p, q) has k-general transmission if and only if it has torsion order k.*

Fix the following notation. For $k = 0$ or $k \geq 2$ and any integer m, let σ^k_m be the permutation exchanging n and $n+1$ for all $n \equiv m \pmod{k}$, and fixing all other integers. Define ι_m as in Remark

3.2. It will be useful to have a formula for the following function.

$$s_{\iota_n \sigma^k_m}(a,b) = \max(a-b+n,0) + \delta\Big[a+n = b \equiv m+1 \pmod{k}\Big]. \tag{8}$$

We use the following fact without proof: for any $\alpha \in \widetilde{\Sigma}_k$, $\operatorname{inv}_k \alpha = 0$ if and only if $\alpha = \iota_n$ for some n, and $\operatorname{inv}_k \alpha = 1$ if and only if $\alpha = \iota_n \sigma^k_m$ for some n, m.

Lemma 4.2. *Suppose (E,p,q) is a genus 1 twice-marked curve with torsion order k, and let $\mathcal{L}$ be a degree d line bundle on E.*

1. *If there exists $m \in \mathbb{Z}$ such that $\mathcal{L} \cong \mathcal{O}_E(mq + (d-m)p)$, then $\tau^{p,q}_{\mathcal{L}} = \iota_{d-1}\sigma^k_{m-1}$.*
2. *If no such m exists, then $\tau^{p,q}_{\mathcal{L}} = \iota_{d-1}$.*

Proof. Riemann-Roch implies that the only special line bundle on a genus 1 curve is $\mathcal{O}_E$. Therefore if $\mathcal{L}$ is not isomorphic to $\mathcal{O}_E(mq+(d-m)p)$ any integer m, then $s^{p,q}_{\mathcal{L}} = s_{\iota_{d-1}}$. This proves part (2).

Now assume $\mathcal{L} \cong \mathcal{O}_E(mq + (d-m)p)$ for some m. Then $\mathcal{L}(ap - bq)$ is special if and only if it has degree 0 and $b \equiv m \pmod{k}$, and it has $h^0 = h^1 = 1$ in that case. Therefore

$$s^{p,q}_{\mathcal{L}}(a,b) = \max\{d-1+a-b, 0\} + \delta\left(b \equiv m \pmod{k} \text{ and } a = b-d+1\right).$$

By Equation (8), this is $s_{\iota_{d-1}\sigma^k_{m-1}}(a,b)$. □

Proof of theorem 4.1. If (E,p,q) is genus 1 and has torsion order k, then Lemma 4.2 shows that $W^{\iota_n}(E,p,q) = \operatorname{Pic}^{n+1}(E)$, $W^{\iota_n\sigma^k_m}(E,p,q)$ is a single point, and $W^\alpha(E,p,q)$ is empty for any other $\alpha \in \widetilde{\Sigma}_k$. So (E,p,q) has k-general transmission.

On the other hand, for any $k' \neq k$, the fact that σ^k_0 occurs as a transmission permutation on (Γ, v, w) implies that (Γ, v, w) does not have k'-general transmission: either $k \nmid k'$ and $\sigma^k_0 \notin \widetilde{\Sigma}_k$, or $k \mid k'$, and $\operatorname{inv}_{k'} \sigma^k_0 = k'/k \geq 2$, but $W^{\sigma^k_0}(E,p,q)$ has codimension 1. □

5. Relative transmission loci

The purpose of this section is to prove the following theorem.

Theorem 5.1. *Let $\mathcal{C}_{g,2}$ denote the locus in $\overline{\mathcal{M}}_{g,2}$ of twice-marked chains, including twice-marked smooth curves. Then $(X,p,q) \mapsto \dim W^\tau(X,p,q)$ is an upper semicontinuous function on $\mathcal{C}_{g,2}$.*

To do so, we formulate a relative version of transmission loci, defined for versal deformations. For simplicity, we have not attempted to define relative transmission loci for more general families, and instead freely make assumptions that will simplify the exposition and be sufficient for our purposes. We freely use various standard facts about deformation theory of nodal curves with marked points; a nice summary of what is needed, with references, can be found in [LO19, p. 20-21].

5.1. **Versal deformation of a chain.** Begin with a single twice-marked chain (X_0, p_0, q_0) of genus g. We will consider a versal deformation of (X_0, p_0, q_0), i.e. a smooth morphism from a base scheme B, which we assume smooth and irreducible, to the stack $\overline{\mathcal{M}}_{g,2}$. Note that here, the subscript 0 refers to a deformation parameter, rather than indexing components in the chain. This amounts to a flat proper morphism $\pi : X \to B$ with two sections $p, q : B \to X$. We denote the members of this family by (X_b, p_b, q_b) for geometric points $b \in B$. Since the universal curve is a smooth stack, the total space X is smooth. The morphism π is not smooth, of course, but after shrinking B if necessary we can identify that non-smooth locus as a disjoint union of ℓ codimension-2 subschemes $Z_1, \cdots Z_{\ell-1}$, corresponding to the nodes of X_0. In each member (X_b, p_b, q_b) of the family, the components Z_i meeting X_b are in bijection with the nodes of X_b, which can each be viewed as a node of X_0 that has not been smoothed as X_0 deforms to X_b. The images $\pi(Z_i)$ are locally principal subschemes in B; shrinking B if necessary we assume that they are principal. Each $\pi^{-1}(\pi(Z_i))$

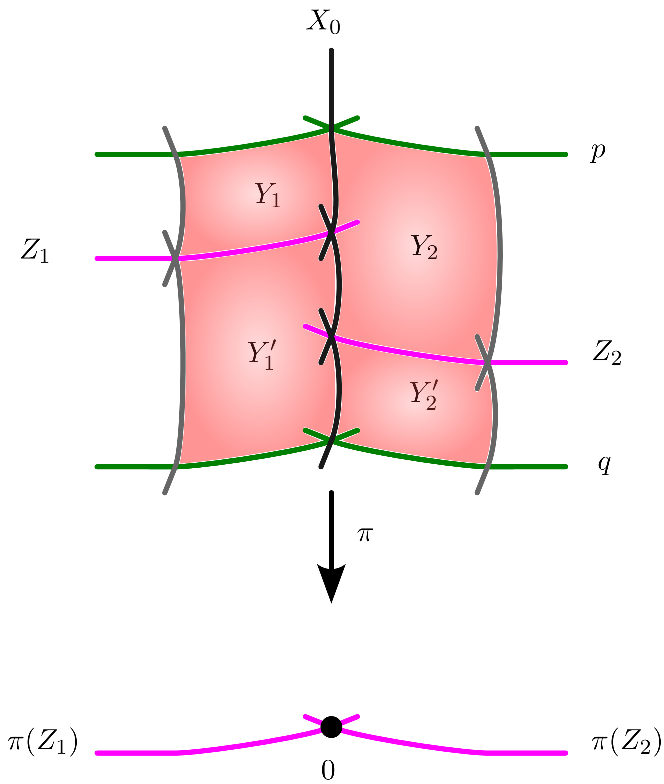

FIGURE 1. A versal deformation of a chain, and the divisors Y_i, Y_i'.

is a principal divisor in X, which can be decomposed into a union of two divisors Y_i, Y_i' meeting transversely at Z_i; we take Y_i to be the "upper half," so that for each $b \in \pi(Z_i)$, $C_b \cap Y_i$ contains p_b, while Y_i' is the "lower half" containing q_b for all such b. See Figure 1.

The divisors Y_i, Y_i' explain why transmission loci were defined the way that they were for chains. Two line bundles $\mathcal{L}, \mathcal{L}'$ on the family X agree on the smooth members if one is obtained from the others via twisting by these divisors. So it is reasonable to require our dimension bounds hold for *all* such twists when working in this family. Now, the line bundles $\mathcal{O}(Y_i)$ restrict to the family members meeting Z_i (i.e. where the ith node has not been smoothed) as follows.

$$\mathcal{O}_X(Y_i)\mid_{Y_i} \cong \mathcal{O}_{Y_i}(-Z_i), \text{ and } \mathcal{O}_X(Y_i)\mid_{Y_i'} \cong \mathcal{O}_{Y_i'}(Z_i). \tag{9}$$

The assumption that $\pi(Z_i)$ is principal means that $\mathcal{O}_X(Y_i') \cong \mathcal{O}_X(-Y_i)$.

5.2. **Relative transmission loci.** This suggests that we ought to define a relative transmission locus for this family as follows. For an $(\ell-1)$-tuple $\vec{n} \in \mathbb{Z}^{\ell-1}$, let $\mathcal{Y}(\vec{n}) = \mathcal{O}_X(\sum_{i=1}^{\ell-1} n_i Y_i)$. Fix a permutation $\tau \in \mathrm{ASP}$ and let $d = \chi_\tau + g$. Denote by $\mathrm{Pic}^{d,p}(\pi) \to B$ the relative Picard scheme of line bundles having degree d on the component of each fiber containing p_b and degree 0 on every other component (as in [LO19, Notation 3.2.4]), and let $\rho : X \times_B \mathrm{Pic}^{d,p}(\pi) \to \mathrm{Pic}^{d,p}(\pi)$ be the projection. Let $\mathcal{U}$ be a Poincaré line bundle on $X \times_B \mathrm{Pic}^{d,p}(\pi)$. The points of $\mathrm{Pic}^{d,p}(\pi)$ are pairs $(b, [\mathcal{L}])$ of a point $b \in B$ and a (isomorphism class of a) line bundle $\mathcal{L}$ on X_b of the prescribed multidegree. Define a subscheme of $\mathrm{Pic}^{d,p}(\pi)$ as follows.

$$W^\tau(\pi,p,q) = \Big\{ x \in \mathrm{Pic}^{d,p}(\pi) : \quad \dim \left(R^1\rho_* \mathcal{U}\left(ap - bq\right) \otimes \mathcal{Y}(\vec{n}) \right)_x \geq s_{\tau^{-1}}(b, a+1) \quad \text{for all } a,b \in \mathbb{Z},\ \vec{n} \in \mathbb{Z}^{\ell-1} \Big\}.$$

Here we abuse notation slightly and write $p, q, \mathcal{Y}(\vec{n})$ for the pullbacks of divisors and line bundles on X to $X \times_B \mathrm{Pic}^{d,p}$. For each choice of $a, b \in \mathbb{Z}, \vec{n} \in \mathbb{Z}^{\ell-1}$, the locus of $x \in \mathrm{Pic}^{d,p}(\pi)$ satisfying the inequality above has the natural structure of a closed subscheme, cut out by a *Fitting ideal* of the sheaf $R^1\rho_*\mathcal{U}\left(ap - bq)\right) \otimes \mathcal{Y}(\vec{n})$. Therefore $W^\tau(\pi,p,q)$ is a closed subscheme.

This construction will be no good unless its fibers over smooth members (X_b, p_b, q_b) coincide with the construction of $W^\tau(X_b, p_b, q_b)$. Fortunately, they do. Since the fibers of ρ are 1-dimensional, the theorem on cohomology and base change implies that for all $t \in B$, the fiber $W^\tau(\pi,p,q)_t$ over t is

$$\Big\{ [\mathcal{L}] \in \mathrm{Pic}^{d,p_t}(X_t) : h^1(X_t, \mathcal{L}(ap_t - bq_t) \otimes \mathcal{Y}(\vec{n})_t) \geq s_{\tau^{-1}}(b, a+1) \text{ for all } a, b \in \mathbb{Z}, \vec{n} \in \mathbb{Z}^{\ell-1} \Big\}, \tag{10}$$

and when X_t is smooth, we have $\mathcal{Y}(\vec{n})_t \cong \mathcal{O}_{X_t}$ for all $\vec{n} \in \mathbb{Z}^{\ell-1}$. It follows that if X_t is smooth, then the fiber $W^\tau(\pi,p,q)_t$ is none other than $W^\tau(X_t, p_t, q_t)$, as we would hope.

Remark 5.2. A word on our use of cohomology and base change may clarify why we use h^1 rather than h^0 here. The fact we have used is that if $f : X \to Y$ is a morphism, and $\mathcal{F}$ is a sheaf on X, flat over Y, *such that* $H^2(X_y, \mathcal{F}_y) = 0$ *for all* $y \in Y$ (e.g. if the fibers are one-dimensional as in our situation), then the natural map $\phi^1(y) : (R^1 f_* \mathcal{F})_y \to H^1(X_y, \mathcal{F}_y)$ is an isomorphism. See [Har77, Theorem 12.11], whose notation we mimic in this remark.

On the other hand, when X_t is not smooth, the line bundles $\mathcal{Y}(\vec{n})_t$ on X_t are determined up to isomorphism by Equation (9). Importantly, these bundles are completely determined by the nodal curve X_t itself, not anything about the geometry of the family. In fact, upon restricting Equation (9) to a single fiber X_t, we see that the bundle $\mathcal{O}_X(Y_i) \mid_{X_t}$ is either one of the line bundles $\mathcal{Y}_j$ as in Definition 3.3 if Z_i meets X_t, or $\mathcal{O}_{X_t}$ otherwise, and every node of X_t corresponds to one of the Z_i.

Corollary 5.3. *For all $t \in B$, the fiber $W^\tau(\pi,p,q)_t$ is isomorphic to $W^\tau(X_t, p_t, q_t)$.*

Semicontinuity of fiber dimension now proves Theorem 5.1.

6. Permutations for Brill–Noether varieties and splitting loci

We now demonstrate that classical Brill–Noether loci and Hurwitz–Brill–Noether splitting loci are special cases of transmission loci, with the same expected codimensions. To do so, we first make a simplification to our description of transmission loci.

6.1. **The essential set.** In principle, our definition of transmission loci involve infinitely many inequalities. However, most of them are redundant.

Definition 6.1 ([Pfl22, Definition 7.6])**.** For any $\alpha \in \mathrm{ASP}$, the *essential set* of α is

$$\mathrm{Ess}(\alpha) = \{(a,b) \in \mathbb{Z}^2 : \alpha^{-1}(a-1) \geq b > \alpha^{-1}(a),\ \alpha(b-1) \geq a > \alpha(b)\}$$

This definition mirrors the "essential set" defined in [Ful92] for degeneracy loci in a finite-dimensional vector space, and plays the same role in our analysis. Its key property is the following.

Proposition 6.2 ([Pfl22, Corollary 7.9])**.** *Say that a permutation α has* bounded difference *if $\{|\alpha(n) - n| : n \in \mathbb{Z}\}$ is bounded. If $\alpha, \beta \in \mathrm{ASP}$ have the same shift and α has bounded difference, then $\alpha \leq \beta$ if and only if $s_\alpha(a,b) \leq s_\beta(a,b)$ for all $(a,b) \in \mathrm{Ess}(\alpha)$.*

Corollary 6.3. *For any twice-marked chain (X,p,q) of genus g, and permutation τ of bounded difference,*

$$W^\tau(X,p,q) = \Big\{ [\mathcal{L}] \in \operatorname{Pic}^{\chi_\tau + g}(C) : h^0\big(C, \mathcal{L}((a-1)p - bq)\big) \geq s_\tau(a,b) \text{ for all } (a,b) \in \operatorname{Ess}(\tau) \Big\}.$$

That is, we need only bound h^0 for values of (a,b) in the essential set.

6.2. **Brill–Noether loci.** For positive integers g,r,d and a genus g curve C, the classical Brill–Noether locus $W^r_d(C)$ is defined to be $\{[\mathcal{L}] \in \operatorname{Pic}^d(C) : h^0(C,\mathcal{L}) \geq r+1\}$. We are interested only in cases where $r+1, g-d+r \geq 1$, since otherwise $W^r_d(C) = \operatorname{Pic}^d(C)$. In these cases, the expected codimension of $W^r_d(C)$, as predicted e.g. by Porteous's formula, is $(r+1)(g-d+r)$. Corollary 6.3 provides a route to identify Brill–Noether loci with transmission loci; we need only specify the right permutation.

Definition 6.4. Suppose $r \geq \max\{0, \chi+1\}$. Let γ^r_χ be the unique permutation that restricts to the unique *increasing* bijection between the following sets. For readability, we use interval notation, but in each case we mean the intersection with $\mathbb{Z}$.

$$\begin{aligned} (-\infty,-1] &\xrightarrow{\sim} (-\infty,-r-1] \cup [1, r-\chi] \\ [0,\infty) &\xrightarrow{\sim} [-r,0] \cup [r-\chi+1,\infty) \end{aligned}$$

The following facts about γ^r_χ are straightforward to verify from definitions.

Lemma 6.5. *The essential set of $\gamma = \gamma^r_\chi$ is $\{(1,0)\}$, and $s_\gamma(1,0) = r+1$. The shift of γ is χ. The set of inversions of γ is $([-(r-\chi),-1] \times [0,r]) \cap \mathbb{Z}^2$.*

Corollary 6.6. *If (C,p,q) is a genus g smooth twice-marked curve, and r,d are integers with $r \geq 0$ and $g-d+r > 0$, then $W^r_d(C) = W^{\gamma^r_{d-g}}(C,p,q)$, and $\operatorname{inv}(\gamma^r_{d-g}) = (r+1)(g-d+r)$. In particular, if (C,p,q) has 0-general transmission, then it is Brill-Noether general, in the sense that all $W^r_d(C)$ have dimension exactly $g-(r+1)(g-d+r)$.*

6.3. **Hurwitz–Brill–Noether splitting loci.** Throughout this section, fix an integer $k \geq 2$. As in [LLV25], let $\mathcal{H}_{g,k}$ denote the Hurwitz space, parameterizing degree-k covers $\pi : C \to \mathbb{P}^1$ from a genus g smooth curve, and let $\mathcal{H}_{g,k,2}$ denote the moduli space of degree-k covers together with two marked points $p,q \in C$ of total ramification. Equivalently, this is the moduli space of twice-marked smooth curves (C,p,q) such that $kp \sim kq$. Hurwitz–Brill–Noether theory concerns the description of the Brill–Noether varieties of a general point in $\mathcal{H}_{g,k}$, and early work concerned the determination of $\dim W^r_d(C)$ such curves [CM99, CM02, Pfl17a, JR21]. More recently, Cook-Powell–Jensen [CPJ22a, CPJ22b] and Larson [Lar21a] independently refined the theory by observing that for a point in $\mathcal{H}_{g,k}$, $\operatorname{Pic}(C)$ has a much more refined and well-behaved stratification: into *splitting loci.* In particular, this refinement is naturally studied by Larson's results on splitting loci [Lar21b]. Splitting loci answered a riddle originally present in the formula for $\dim W^r_d(C)$ conjectured in [Pfl17a] and proved in [JR21], which suggested that $W^r_d(C)$ is not equidimensional for general k-gonal curves; the reason is that $W^r_d(C)$ decomposes into several "balanced splitting loci" that may have different dimensions. The full suite of classical theorems in Brill–Noether theory has recently been generalized to the context of splitting loci by Larson, Larson, and Vogt [LLV25].

Definition 6.7. A *splitting type* is a nondecreasing k-tuple $\vec{e} = (e_1, \cdots, e_k) \in \mathbb{Z}^k$. For a splitting type $\vec{e}$, let $x_{\vec{e}} : \mathbb{Z} \to \mathbb{N}$ be the function $x_{\vec{e}}(m) = \sum_{n=1}^k \max\{e_n + 1 + m, 0\}$. For a cover $\pi : C \to \mathbb{P}^1$ in $\mathcal{H}_{g,k}$ with $P = \pi^* \mathcal{O}_{\mathbb{P}^1}(1)$, define the *splitting locus*

$$W^{\vec{e}}(C,P) = \{[\mathcal{L}] \in \operatorname{Pic}^{d(\vec{e})}(C) : \ h^0(\mathbb{P}^1, \mathcal{L}(mP)) \geq x_{\vec{e}}(m) \text{ for all } m \in \mathbb{Z}\}.$$

The *expected codimension* of $W^{\vec{e}}(C,P)$ is the number $u(\vec{e}) = \sum_{0 \leq m,n < k} \max\{0, e_m - e_n - 1\}$.

Our notation differs slightly from [LLV25], in that we specify the divisor class P. If k is the gonality of C this can be left implicit, but we include it for emphasis and because for larger k the choice of P need not be unique. The definition of splitting loci above is reminiscent of our definition of transmission loci, and indeed this is not a coincidence: we show in this section that for (C,p,q) in $\mathcal{H}_{g,k,2}$, i.e. in the situation where $P \sim kp \sim kq$, splitting loci *are* transmission loci, and the stratification by transmission loci may be viewed as a further refinement of the splitting type stratification. There is another, more geometrically meaningful definition of splitting loci that explains their name: $[\mathcal{L}]$ belongs to the open part of $W^{\vec{e}}(C,P)$ (the complement of all $W^{\vec{f}}(C,P) \subsetneq W^{\vec{e}}(C,P)$) if and only if $\pi_*\mathcal{L}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(e_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(e_k)$.

Larson proved in [Lar21a], via intersection theory techniques developed in [Lar21b], that for *every* point in $\mathcal{H}_{g,k}$, $W^{\vec{e}}(C,P)$ is nonempty if $u(\vec{e}) \leq g$, and every component has codimension at most $u(\vec{e})$ if so. For a *general point* in $\mathcal{H}_{g,k}$, Larson [Lar21a] and Cook-Powell–Jensen [CPJ22a] independently proved that every component of $W^{\vec{e}}(C,P)$ has codimension *at least* $u(\vec{e})$; the results of this paper provide a new proof of that in the broader context of transmission loci. Much stronger results about irreducibility, smoothness, and monodromy of splitting loci can be found in [LLV25], and we conjecture that the same results should hold for transmission loci (Conjecture 7.2).

6.4. **Associating permutations to splitting loci.** For a point (C,p,q) of $\mathcal{H}_{g,k,2}$, we have $kp \sim kq$ and therefore every transmission permutation is in $\widetilde{\Sigma}_k$. There is a cover $\pi : C \to \mathbb{P}^1$ totally ramified at p and q, so the class of the fiber is $P = kp$. In particular, for all $m \in \mathbb{Z}$, $h^0(C, \mathcal{L}(mP)) = s^{p,q}_{\mathcal{L}}(1+mk, 0) = s^{p,q}_{\mathcal{L}}(1+ak, bk)$ for all $a,b \in \mathbb{Z}$ such that $a-b=m$. So the transmission function/permutation of $\mathcal{L}$ determines its splitting type. We demonstrate in this section how to read the splitting type from a permutation in $\widetilde{\Sigma}_k$, and how to identify a splitting locus with a transmission locus. The content of this section is not novel, and indeed the affine symmetric groups are used systematically to study splitting loci in [LLV25] (see especially Theorem 1.4); we include this section only for completeness and to explain the story in our notation.

Call a permutation $\rho \in \widetilde{\Sigma}_k$ *residual* if it restricts to a permutation of $\{0,\ldots,k-1\}$. For any $\alpha \in \widetilde{\Sigma}_k$, there exists a unique pair (ρ,π) of a residual permutation and a *k-periodic* function such that

$$\alpha(n) = \rho(n) + 1 + k\pi(n).$$

Conversely, every such pair (ρ,π) gives an $\alpha \in \widetilde{\Sigma}_k$. This decomposition is convenient for studying the splitting type; the $+1$ above is a convenience, as we will see. If α has this decomposition, then its inverse is

$$\alpha^{-1}(n) = \rho^{-1}(n-1) - k\pi(\rho^{-1}(n-1)).$$

The residual permutation is irrelevant to the values of the slipface determining the splitting type of α, because

$$\begin{aligned} s_\alpha(1+ak, bk) &= \sum_{n=0}^{k-1} \#\{q \in \mathbb{Z} : n+qk \geq bk \text{ and } \alpha(n+qk) < 1+ak\} \\ &= \sum_{n=0}^{k-1} \max\{0, a-b-\pi(n)\}. \end{aligned}$$

So the splitting type of α is given by sorting the tuple $(-\pi(0)-1, \cdots, -\pi(k-1)-1)$ to nondecreasing order. In particular, this shows that every splitting type occurs for some $\alpha \in \widetilde{\Sigma}_k$. We may also obtain a useful bound on $\mathrm{inv}_k(\alpha)$ from π alone. Observe that

$$\mathrm{inv}_k(\alpha) \leq \# \left\{ (m,n) : 0 \leq n < k,\ \left\lfloor \frac{m}{k} \right\rfloor < \left\lfloor \frac{n}{k} \right\rfloor,\ \left\lfloor \frac{\alpha(m)-1}{k} \right\rfloor < \left\lfloor \frac{\alpha(n)-1}{k} \right\rfloor \right\},$$

and equality holds if and only if α is increasing on $\{0, \cdots, k-1\}$ and α^{-1} is increasing on $\{1, \cdots, k\}$. This upper bound may be computed from π alone. For fixed $0 \leq m, n < k$, consider which pairs $(m - qk, n)$ are counted by this upper bound. We have $\left\lfloor \frac{m-qk}{k} \right\rfloor < \left\lfloor \frac{n}{k} \right\rfloor$ if and only if $0 < q$, and $\left\lfloor \frac{\alpha(m-qk)-1}{k} \right\rfloor < \left\lfloor \frac{\alpha(n)-1}{k} \right\rfloor$ if and only if $\pi(m) - q < \pi(n)$. So the number of such pairs is $\max\{0, \pi(n) - \pi(m) - 1\}$, and we deduce that

$$\operatorname{inv}_k(\alpha) \leq \sum_{0 \leq m,n < k} \{\max\{0, \pi(n) - \pi(m) - 1\}, \tag{11}$$

with equality if and only if α is increasing on $\{0, \cdots, k-1\}$ and α^{-1} is increasing on $\{1, \cdots, k\}$.

Conveniently, this equality case coincides with another useful situation: when the essential set consists only of pairs $(1 + ak, bk)$. In this case, $W^\alpha(C, p, q)$ is identical to a splitting locus. Note that we use here the fact that if α is increasing on a set, then it is automatically increasing on any translate of that set by a multiple of k. We will now classify the choices of ρ, π for which this situation occurs. Observe that if $0 \leq m, n < k$, and we wish to determine which of $\alpha(m), \alpha(n)$ is larger, we can do so by comparing $\pi(m), \pi(n)$, and breaking a tie with $\rho(m), \rho(n)$. From this and the discussion above, we obtain the following classification.

Lemma 6.8. *Say that ρ is* increasing when π is tied *if for all $0 \leq m, n < k$, if $\pi(m) = \pi(n)$ then $m < n$ if and only if $\rho(m) < \rho(n)$. The permutation α defined above is increasing on $\{0, \cdots, k-1\}$ if and only if $\pi(0) \leq \cdots \leq \pi(k-1)$ and ρ is increasing when π is tied. On the other hand, α^{-1} is increasing on $\{1, \cdots, k\}$ if and only if $\pi(\rho^{-1}(0)) \geq \pi(\rho^{-1}(1)) \geq \cdots \geq \pi(\rho^{-1}(k-1))$ and ρ is increasing when π is tied.*

Corollary 6.9. *For a fixed k-periodic function π, there exists a residual permutation ρ such that $\alpha = \rho + 1 + k\pi$ is increasing on $\{0, \cdots, k-1\}$ and α^{-1} is increasing on $\{1, \cdots, k\}$ if and only if π is nondecreasing on $\{0, \cdots, k-1\}$. If such ρ exists, it is unique.*

Proof. The lemmas above show that it is necessary for π to be nondecreasing on $\{0, \cdots, k-1\}$. If so, then α has the desired property if and only if ρ is increasing when π is tied, and precomposing with ρ^{-1} reverses the order of $\pi(0), \cdots, \pi(k-1)$. This uniquely determines ρ: $\rho(0), \cdots, \rho(k-1)$ must consist of the indices for which $\pi(n)$ is maximum, in increasing order, followed by the indices where $\pi(n)$ is the second-larger value, in increasing order, and so on. Explicitly,

$$\rho(n) = \#\{m : 0 \leq m < k,\ \pi(m) < \pi(n)\} + \#\{m : 0 \leq m < n,\ \pi(m) = \pi(n)\}$$

for all $0 \leq n < k$. Note that this formula closely resembles the construction of the permutation $w(\vec{e})$ in [LLV25, Theorem 1.4]; up to some conventions, the two constructions are the same. ☐

Definition 6.10. For any splitting type (nondecreasing k-tuple) $\vec{e} = (e_1, \cdots, e_k)$, let $\gamma_{\vec{e}} = \rho + 1 + k\pi$, where π is the k-periodic function determined by $\pi(n) = -e_{k-n} - 1$ for $0 \leq n < k$ (so that $\pi(0) \leq \cdots \leq \pi(k-1)$) and ρ is the permutation described in Corollary 6.9.

Proposition 6.11. *The permutation $\gamma = \gamma_{\vec{e}}$ has $s_\gamma(1 + ak, bk) = x_{\vec{e}}(a - b)$ for all $a, b \in \mathbb{Z}$, $\operatorname{Ess}(\gamma) \subseteq \{(1 + ak, bk) :\ a, b \in \mathbb{Z}\}$, and $\operatorname{inv}_k(\gamma) = u(\vec{e})$.*

Proof. The computation of $s_\gamma(1 + ak, bk)$ is carried out above. The description of γ in Corollary 6.9 and identity $\gamma(n + k) = \gamma(n) + k$ implies that $\gamma(b) < \gamma(b-1)$ is only possible for $b \equiv 0 \pmod k$ and $\gamma^{-1}(a) < \gamma^{-1}(a-1)$ is only possible for $a \equiv 1 \pmod k$, hence $(a, b) \in \operatorname{Ess}(\gamma)$ implies that $a \equiv 1 \pmod k$ and $b \equiv 0 \pmod k$. Equation (11) implies that $\operatorname{inv}_k(\gamma) = u(\vec{e})$. ☐

Corollary 6.12. *For any point (C, p, q) of $\mathcal{H}_{g,k,2}$, the splitting locus $W^{\vec{e}}(C, kp)$ is equal to the transmission locus $W^{\gamma_{\vec{e}}}(C, p, q)$, and the expected codimensions match. In particular, if (C, p, q) has*

k-general transmission, then it is Hurwitz–Brill–Noether general, *in the sense that all its splitting loci have the expected dimension.*

7. Questions and conjectures

We end with some conjectures and questions for future work. We mentioned the first conjecture in the introduction.

Conjecture 7.1. *Let $k \geq 2$. On* any *(C, p, q) with $kp \sim kq$, every component of $W^\tau(C, p, q)$ has codimension* at most $\mathrm{inv}_k(\alpha)$.

After being circulated in an early draft of this paper, Conjecture 7.1 was proved by Daksh Aggarwal, along with other structural results on transmission loci.

Conjecture 7.2. *All statements of* [LLV25, Theorem 1.2] *generalize from splitting loci to transmission loci of a general point in $\mathcal{H}_{g,k,2}$ (or $\mathcal{M}_{g,2}$, for $k = 0$). In particular, the intersection theory class of $W^\tau(C, p, q)$ is*

$$[W^\tau(C, p, q)] = \frac{N(\tau)}{inv_k(\tau)!} \Theta^{\mathrm{inv}_k(\tau)},$$

where $N(\tau)$ is the number of reduced words for $\iota_{-\chi_\tau}\tau$ (this $\iota_{-\chi_\tau}$ serves to convert τ to something of shift 0). Furthermore, at any *point of $\mathcal{H}_{g,k,2}$, $W^\tau(C, p, q)$ supports this intersection class.*

In the case $k = 0$, this enumerative formula follows from the results of [Pfl25], so the $k \geq 2$ case is of primary interest. It is very plausible to me that the methods of [LLV25] can be adapted to prove this conjecture. A notable special case is $\mathrm{inv}_k(\tau) = g$; as observed in Example 3.8 the Demazure product machinery gives a bijection between reduced words for τ and points of $W^\tau(X, p, q)$ on an elliptic chain X, and more generally Proposition 3.7 points to the very explicit link between reduced words and transmission loci on elliptic chains that is provided by the Demazure product.

Conjecture 7.3. *Let $\pi : X \to B$ be a versal family in $\mathcal{M}_{g,2}$ (if $k = 0$) or $\mathcal{H}_{g,k,2}$ (if $k \geq 2$). In a relative transmission locus $W^\tau(\pi, p, q)$, every component has codimension at most* $\mathrm{inv}_k(\tau)$.

The importance of this conjecture is that it would allow a "regeneration theorem," akin to the regeneration theorem for limit linear series, to be proved for transmission loci: if a transmission locus has the expected dimension on a singular curve, then this conjecture would show that it is part of a component that also lies over nearby smooth curves with the expected dimension.

Question 7.4. For which permutations τ and genera g does there exist a smooth twice-marked curve (C, p, q) of genus g and line bundle $\mathcal{L}$ with $\tau^{p,q}_{\mathcal{L}} = \tau$?

I expect this question to be quite hard to answer in full generality, since it strictly generalizes the question of which Weierstrass semigroups occur on marked algebraic curves, which is still wide open (see e.g. [KY13]). The last question is a variation that adds an expected dimension requirement, similar to the study of dimensionally proper Weierstrass points in [EH87], for example.

Question 7.5. Given $\tau \in \widetilde{\Sigma}_k$ and a genus $g < \mathrm{inv}_k(\tau)$, suppose that (C, p, q) and $\mathcal{L}$ are such that $W^\tau(C, p, q)$ is nonempty. Call a point of $W^\tau(C, p, q)$ *dimensionally proper* if, in a versal deformation $\pi : X \to B$ of (C, p, q) in $\mathcal{M}_{g,2}$ (if $k = 0$) or $\mathcal{H}_{g,k,2}$ (if $k \geq 2$), $(C, p, q, [\mathcal{L}])$ belongs to a component of $W^\tau(\pi, p, q)$ of dimension exactly the expected dimension $\dim B + g - \mathrm{inv}_k(\tau)$. For which τ, g, do there exist such dimensionally proper points?

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