

On linear series with negative Brill-Noether number

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Abstract

Brill-Noether theory studies the existence and deformations of curves in projective spaces; its basic object of study is $\mathcal{W}_{d,g}^r$, the moduli space of smooth genus g curves with a choice of degree d line bundle having at least $(r+1)$ independent global sections. The Brill-Noether theorem asserts that the map $\mathcal{W}_{d,g}^r \rightarrow \mathcal{M}_g$ is surjective with general fiber dimension given by the number $\rho = g - (r+1)(g-d+r)$, under the hypothesis that $0 \leq \rho \leq g$. One may naturally conjecture that for $\rho < 0$, this map is generically finite onto a subvariety of codimension $-\rho$ in \mathcal{M}_g . This conjecture fails in general, but seemingly only when $-\rho$ is large compared to g . This paper proves that this conjecture does hold for at least one irreducible component of $\mathcal{W}_{d,g}^r$, under the hypothesis that $0 < -\rho \leq \frac{r}{r+2}g - 3r + 3$. We conjecture that this result should hold for all $0 < -\rho \leq g + C$ for some constant C , and we give a purely combinatorial conjecture that would imply this stronger result.

1 Introduction

Throughout this paper, a *curve* will always mean a complete algebraic curve over \mathbf{C} , with at worst nodes as singularities.

Brill-Noether theory studies the ways curves can lie in projective spaces. One of the principle objects of study is the moduli space $\mathcal{W}_{d,g}^r$, which parameterizes curves of genus g together with a chosen line degree d line bundle with at least $(r+1)$ independent global sections. The geometry of this space is well-understood over general curves; in particular the Brill-Noether theorem [7] states that when $r \geq 0$ and $g-d+r \geq 1$ a general fiber of the map $\mathcal{W}_{d,g}^r \rightarrow \mathcal{M}_g$ is either empty or has dimension given by the *Brill-Noether number*, traditionally denoted ρ and defined as follows.

$$\rho(g, d, r) = g - (r+1)(g-d+r)$$

Furthermore, the general fiber is empty if and only if $\rho \geq 0$. This paper considers the extension of the Brill-Noether theorem to the case $\rho < 0$, i.e. to non-general curves. The main result is the following.

Theorem 1.1. *Suppose that g, d, r are positive integers with $g-d+r \geq 2$ and $0 > \rho \geq -\frac{r}{r+2}g + 3r - 3$. Then $\mathcal{W}_{d,g}^r$ has a component of dimension $\dim \mathcal{M}_g + \rho$, whose image in \mathcal{M}_g has codimension equal to $-\rho$, and whose general member has rank exactly r .*

The assumptions $r \geq 1$ and $g-d+r \geq 2$ are necessary: if $g-d+r \leq 0$ then $\rho > 0$, while if $g-d+r = 1$ or $r = 0$ (these situations are dual to each other) then $\mathcal{W}_{d,g}^r$ is empty if $\rho < 0$.

Note that $\dim \mathcal{M}_g + \rho$ is a lower bound on the dimension of any component of $\mathcal{W}_{d,g}^r$ (we will see one proof in section 3, by combining formula 1 and lemma 3.5), so this theorem asserts that this bound is achieved for not-too-negative values of ρ .

The proof proceeds by induction on the genus. The statement of 1.1 is not suitable for induction; we instead introduce the notion of *twisted Weierstrass points*, and prove a suitable generalization in this context. The method is based on the limit linear series techniques introduced by Eisenbud and Harris [4] to construct certain Weierstrass points. As a second application of our techniques, we also prove that the naive dimension

estimate for the number of moduli of a Weierstrass point always fails when the Semigroup does not satisfy a combinatorial condition called primitivity (Theorem 7.2).

The outline of this paper is as follows. Section 2 discusses background and previous results, and states some conjectures. Section 3 introduced the notion of a *twisted Weierstrass point* corresponding to a partition P ; we define moduli spaces $\mathcal{W}_g(P)$ of twisted Weierstrass points on genus g curves and show that studying \mathfrak{g}_d^r s on genus g curves is equivalent to studying twisted Weierstrass points on genus g curves corresponding to the “box-shaped” partition $((g - d + r)^{r+1})$. Section 5 describes a construction, using the theory of limit linear series, of twisted Weierstrass points in genus $g + 1$ from twisted Weierstrass points in genus g . Section 6 defines a combinatorial invariant called the *difficulty* of a partition, and shows how bounding this invariant implies the existence of dimensionally proper twisted Weierstrass points. Sections 7 and 8 demonstrate this technique by bounding the difficulty of two different sorts of partitions. Section 7 reproves a theorem of Eisenbud and Harris on dimensionally proper Weierstrass points¹, and then proves theorem 7.2, showing that a primitivity hypothesis in that theorem cannot be removed. Finally, section 8 gives a bound on the difficulty of box-shaped partitions sufficient to prove theorem 1.1.

2 Background and conjectures

As the numbers g, d, r vary (constrained by $g - d + r \geq 2$), the spaces $\mathcal{W}_{d,g}^r$ exhibit two very different sorts of behavior. For $0 \leq \rho \leq g$, the situation is well-understood: $\mathcal{W}_{d,g}^r$ is irreducible, maps subjectively to \mathcal{M}_g , and has general fiber of dimension ρ . On the other hand, when $-\rho \gg 0$ the dimension estimate $(3g - 3) + \rho$ fails dramatically. Indeed, many natural families of curves (such as complete intersections, determinantal curves, and curves on rational surfaces) have degree and genus such that ρ is extremely negative, and yet these families have rather large dimension. This phenomenon, observed in numerous examples, has led to the following folklore conjecture, sometimes called the *rigid curves conjecture*.²

Conjecture 2.1. For all r , there is a positive number $C(r)$ such that whenever $\mathcal{W}_{d,g}^r$ is nonempty, all of its components have dimension at least $C(r)g$.

Observe that since $\dim \mathcal{M}_g$ is of course $3g - 3$, and a genus g curve has a g -dimensional space of line bundles of degree d , the dimension of $\mathcal{W}_{d,g}^r$ is always less than $4g$. So another way to state this conjecture is the following: for there is a positive number $C(r)$ such that whenever $\mathcal{W}_{d,g}^r$ is nonempty,

$$C(r) < \frac{\dim \mathcal{W}_{d,g}^r}{g} < 4$$

(and the same is true for each irreducible component of $\mathcal{W}_{d,g}^r$).

This conjecture predicts that there is a sort of “phase transition” as ρ moved from slightly negative values to very negative values, where the Brill-Noether dimension estimate begins to fail and the natural tendency of embedded curves to vary in families of dimension linear in g (in addition to the $\dim PGL_{r+1}$ degrees of freedom from projective space itself) begins to dominate. The question this paper aims to address is: *where does this phase transition occur?*

Anecdotal evidence suggests that the transition occurs at a constant multiple of g . For example, the simplest case of an embedded curve violating the Brill-Noether dimension estimate is the complete intersection of a quadric and a quartic surfaces in \mathbf{P}^3 . In this case, $(g, d, r) = (9, 8, 3)$ so $\rho = -7$ and the expected dimension of $\mathcal{W}_{8,9}^3$ is 17, but an elementary calculation shows that in fact $\dim \mathcal{W}_{8,9}^3 = 18$. So this counterexample occurs at $\rho = -g + 2$.

Eisenbud and Harris [5] proved that when $\rho = -1$, the space $\mathcal{W}_{d,g}^r$ is irreducible of the expected dimension, and that its image in \mathcal{M}_g is a divisor. Edidin [3] showed that in the case $\mathcal{W}_{d,g}^r$ has all components of the expected dimension, mapping finitely to \mathcal{M}_g . Eisenbud and Harris claimed in their initial paper on limit linear series [6] a result of the same form as our theorem 1.1 was forthcoming, but never published a proof.

¹The proof is essentially the same as theirs, although we circumvent the analysis of limit canonical series.

²This conjecture is usually phrased in terms of components of the Hilbert scheme, but this form is essentially the same.

Our theorem 1.1 gives further evidence that the phase transition occurs in the vicinity of $\rho = -g$. Its main defect is that it only asserts the existence of some component of $\mathcal{W}_{d,g}^r$ that behaves as expected. The reason for this restriction is that our method of proof proceeds by smoothing certain reducible curves; this method cannot detect components of $\mathcal{W}_{d,g}^r$ whose images in \mathcal{M}_g are compact.

Our theorem 7.2 is the first step towards answering a different but very analogous question about Weierstrass points. Since non-primitive semigroups occur in every genus with weights as low as roughly $\frac{1}{2}g$, this shows that the analogous phase transition for Weierstrass points seems to occur when the expected codimension is roughly $\frac{1}{2}g$. We elaborate considerably on the analogous questions for Weierstrass points in [13].

We conclude this section with a general conjecture uniting questions about $\mathcal{W}_{d,g}^r$ with questions about Weierstrass points. See the following section for the definition of $\mathcal{W}_g(P)$ and an explanation of how it is related to $\mathcal{W}_{d,g}^r$.

Conjecture 2.2. Let P be a partition and g a positive integer. Let X be any component of $\mathcal{W}_g(P)$, regarded as a subvariety of $\text{Pic}_g^0 \times_{\mathcal{M}_g} \mathcal{M}_{g,1}$. There exist two positive functions $A(r)$ and $B(r)$ of r with $0 < A(r) < B(r) < 4$, such that:

- $\text{codim} X \leq \min(B(r)g, |P|)$.
- If $|P| \leq A(r)g$, then $\text{codim} X = |P|$.

Question 2.3. In the range $A(r)g \leq |P| \leq B(r)g$, is there a purely combinatorial procedure to determine if $\mathcal{W}_g(P)$ has any components of codimension $|P|$?

3 Twisted Weierstrass points

Theorem 1.1 can be deduced from a slightly stronger result about pointed curves. This section defines the relevant notion, that of *twisted Weierstrass points*, and discusses some basic aspects of their moduli and the connection to theorem 1.1.

Every point p on a smooth curve C determines a numerical semigroup called the *Weierstrass semigroup* of the point; it consists of those integers n such that C has a rational function of degree n whose only pole is at p . For all but finitely many points on a given curve C , this semigroup is $\{0, g+1, g+2, \dots\}$; the other points are called *Weierstrass points*. See [2] for history and applications of Weierstrass points. Consider the following generalization.

Let C be a smooth curve, \mathcal{L} a degree 0 line bundle on C , and $p \in C$ a point. The *twisted Weierstrass sequence* of the triple (C, \mathcal{L}, p) is the following set of nonnegative integers.

$$S(C, \mathcal{L}, p) = \{n \in \mathbf{Z}_{\geq 0} : h^0(\mathcal{L}(np)) > h^0(\mathcal{L}((n-1)p))\}$$

In other words, the twisted Weierstrass sequence is the set of possible pole orders at p of rational sections of \mathcal{L} that are regular away from p . In the special case $\mathcal{L} = \mathcal{O}_C$, the twisted Weierstrass sequence is the classical Weierstrass semigroup. By the Riemann-Roch formula, the complement of S has precisely g elements, where g is the genus of C . If twisted Weierstrass sequences are given the obvious partial ordering, then they are upper semi-continuous families; therefore the *general* twisted Weierstrass sequence is simply

$$S = \{g, g+1, g+2, \dots\}.$$

A triple (C, \mathcal{L}, p) with a different sequence is called a *twisted Weierstrass point*.

We can and will describe a twisted Weierstrass sequence using the (equivalent) data of a partition. Namely, the *twisted Weierstrass partition* $P(C, \mathcal{L}, p)$ is given by the multiset $\{(n+g) - s_n\}$ (restricted to positive entries), where the twisted Weierstrass sequence is $s_0 < s_1 < s_2 < \dots$. Alternatively, one can identify twisted Weierstrass sequences with Schubert cycles, which are identified with partitions in the usual way. This connection will be made more explicit after the definition below.

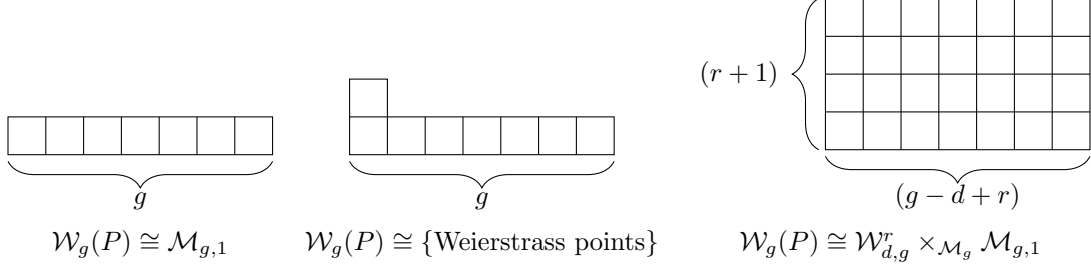


Figure 1: Three examples of partitions and the geometric interpretation of $\mathcal{W}_g(P)$.

Definition 3.1. Given a nonnegative integer g and a partition P , let $\tilde{\mathcal{W}}_g(P)$ denote the moduli space of triples (C, \mathcal{L}, p) , where C is a smooth curve, \mathcal{L} is a line bundle of degree 0 and $p \in C$, such that $P(C, \mathcal{L}, p) = P$. Let $\mathcal{W}_g(P)$ denote the closure of $\tilde{\mathcal{W}}_g(P)$ in $\mathcal{P}ic_g^0 \times_{\mathcal{M}_g} \mathcal{M}_{g,1}$.

The space $\tilde{\mathcal{W}}_g(P)$ can also be described in terms of Schubert cycles, as follows. Any family of curves with marked point and line bundle corresponds to the following data:

- A family of curves $\pi : \mathcal{C} \rightarrow B$,
- A section $s : B \rightarrow \mathcal{C}$, and
- A line bundle \mathcal{L} on \mathcal{C} .

These data determine a filtration of vector bundles on B , given by $E_k = \pi_*(\mathcal{L}((k-1)\Sigma)/\mathcal{L}(-\Sigma))$, where Σ is the divisor given by the image of s . By Grauert's theorem ([9] corollary 12.9), each E_k is a vector bundle of rank k on B . In addition to these, there is also a rank g vector bundle on B given by $F = \pi_*(\mathcal{L}((2g-1)p))$, with an obvious inclusion $F \hookrightarrow E_{2g}$. This inclusion induces a section $t : B \rightarrow G$ to the Grassmannian bundle G of g -planes in E_{2g} . The filtration of E_{2g} given by $E_0 \subset E_1 \subset \dots \subset E_{2g}$ defines, for each partition P , an open Schubert cycle $\tilde{\Sigma}_P \subseteq G$, of codimension $|P|$ (see [8] section 1.5 for a definition and basic properties of Schubert cycles). Then the points of B corresponding to points of $\tilde{\mathcal{W}}_g(P)$ are precisely the inverse image $t^{-1}(\tilde{\Sigma}_P)$.

The description of $\tilde{\mathcal{W}}_g(P)$ in terms of Schubert cycles gives the following bound on its local dimension at any point.

$$\dim_{(C, \mathcal{L}, p)} \tilde{\mathcal{W}}(P) \geq (4g-2) - |P| \quad (1)$$

Definition 3.2. A point $(C, \mathcal{L}, p) \in \tilde{\mathcal{W}}_g(P)$ where equality holds in 1 is called a *dimensionally proper point*.

Example 3.3. Let $P = (g)$. Then $(C, \mathcal{L}, p) \in \tilde{\mathcal{W}}_g(P)$ if and only if $h^0(\mathcal{L}) = 1$ and $h^0(\mathcal{L}(gp)) = 1$. This is true if and only if $\mathcal{L} = \mathcal{O}_C$ and p is not a Weierstrass point. So $\tilde{\mathcal{W}}_g(P)$ is isomorphic to the complement in $\mathcal{M}_{g,1}$ of the locus of Weierstrass points, and $\mathcal{W}_g(P) \cong \mathcal{M}_{g,1}$. Therefore the local dimension at each point is $(3g-2) = (4g-2) - |P|$, so every point is dimensionally proper.

Example 3.4. Let $P = (g \ 1)$. Then $\tilde{\mathcal{W}}_g(P)$ consists of triples (C, \mathcal{L}, p) such that $h^0(\mathcal{L}) = 1$, $h^0((g-1)p) = 1$, and $h^0(gp) = 2$. In other words, this is the locus in $\mathcal{M}_{g,1}$ of simple Weierstrass points. This is étale-locally isomorphic to \mathcal{M}_g , so every point has local dimension $(3g-3) = (4g-2) - |P|$, so all points are dimensionally proper.

We will now study twisted Weierstrass points with the particular type of partition that will be relevant to theorem 1.1. Let $P = (m^n)$ (i.e. the number m occurs n times). This partition corresponds to the following twisted Weierstrass sequence.

$$S = \{g-m, g-m+1, \dots, g-m+n-2, g-m+n-1, g+n, g+n+1, g+n+2, \dots\}$$

Then $(C, \mathcal{L}, p) \in \tilde{\mathcal{W}}_g(P)$ if and only if the following conditions hold.

- $h^0(\mathcal{L}((g-m-1)p)) < h^0(\mathcal{L}(g-m)p) = 1$
- $h^0(\mathcal{L}(g-m+n-1)p) = h^0(\mathcal{L}(g+n-1)p) = n$

These conditions are equivalent to saying that $\mathcal{L}' = \mathcal{L}((g+n-1)p)$ is a line bundle of degree $(g-m+n-1)$ and rank $n-1$, such that p is not a ramification point for either the complete linear series $|\mathcal{L}'|$ or its dual $|\omega_C \otimes \mathcal{L}'^\vee|$. See [1] appendix C for a definition of ramification points, and a proof that there are finitely many of them for a given linear series. Note that this is not true in positive characteristic.

Since any linear series has a finite number of ramification points, this means that for any line bundle \mathcal{M} on C of degree $d = (g-m+n-1)$ and rank $r = (n-1)$, the triple $(C, \mathcal{M}(-dp), p)$ is a point of $\tilde{\mathcal{W}}_g(P)$ for all but finitely many points $p \in C$. The upshot of this is the following.

Lemma 3.5. *Let g, d, r be integers, and let $\tilde{\mathcal{W}}_{d,g}^r \subset \text{Pic}_g^d$ consist of those pairs (C, \mathcal{L}) where \mathcal{L} is a line bundle on C with degree d and $h^0(\mathcal{L}) = r+1$ ³. Let P be the partition $((g-d+r)^{r+1})$. Then there is a map*

$$\begin{aligned} f : \tilde{\mathcal{W}}_g(P) &\rightarrow \tilde{\mathcal{W}}_{d,g}^r \\ (C, \mathcal{L}, p) &\mapsto (C, \mathcal{L}(dp)) \end{aligned}$$

which is surjective, and whose fiber over any point $(C, \mathcal{L}) \in \tilde{\mathcal{W}}_{d,g}^r$ is isomorphic to C with finitely many punctures.

Notice that, in the notion of the lemma, $|P| = (r+1)(g-d+r) = g - \rho(g, d, r)$. It follows from this that the map f in the lemma sends dimensionally proper points to dimensionally proper points. Thus to study dimensionally proper line bundles on curves is equivalent to studying dimensionally proper twisted Weierstrass points given by “box-shaped” partitions. This will be the object of the remainder of the paper.

Remark 3.6. Notice that twisted Weierstrass points have a duality property: namely if P^* is the dual partition of P (that is, $P_n^* = |\{m : P_m > n\}|$), then $\tilde{\mathcal{W}}_g(P) \cong \tilde{\mathcal{W}}(P^*)$, via the map $(C, \mathcal{L}, p) \mapsto (C, \omega_C(-(2g-2)p) \otimes \mathcal{L}^\vee, p)$. This generalizes the fact that $\tilde{\mathcal{W}}_{d,g}^r \cong \tilde{\mathcal{W}}_{(2g-2)-d,g}^{g-d+r-1}$ via $\mathcal{L} \mapsto \omega_C \otimes \mathcal{L}^\vee$ (via the correspondence discussed above), since the dual partition of $((g-d+r)^{r+1})$ is $((r+1)^{g-d+r})$. This duality is reflected, for example, in the two perspectives by which one typically studies classical Weierstrass points: in terms of pole order of rational functions or in terms of ramification of the canonical series.

Question 3.7. Let $\mu(P, g)$ be the maximum codimension of a component of $\mathcal{W}_g(P)$ (or $-\infty$ if there are none). When is $\mu(P, g) < |P|$? Is there a purely combinatorial description of which partitions P and integers g give strict inequality?

We will define in section 6 a function $\delta(P)$ of partitions such that $\mu(P, g) = |P|$ whenever $g \geq \frac{1}{2}(|P| + \delta(P))$. Bounding this function will give theorem 1.1. First we describe the smoothing argument which underlies the definition of $\delta(P)$.

4 Limits of twisted Weierstrass points

To prove the existence of certain twisted Weierstrass points, it will be necessary to allow the curves to degenerate to singular curves, and to have a suitable notion of limits of the twisted Weierstrass points. Such a notion is provided by limit linear series, as introduced by Eisenbud and Harris [6]. We begin by recalling the relevant definitions; see [11] for an expository treatment.

A *linear series* of degree d and rank r on a smooth curve C , also called a \mathfrak{g}_d^r , is a pair $L = (\mathcal{L}, V)$, where \mathcal{L} is a degree d line bundle and $V \subseteq H^0(\mathcal{L})$ is an $(r+1)$ -dimensional vector space of sections. The moduli space of genus g curves with a chosen \mathfrak{g}_d^r is denoted $\mathcal{G}_{d,g}^r$. Given a linear series L and a point $p \in C$, the *vanishing sequence* of L at p is the set of integers n such that V contains a section vanishing to order exactly n at p .

³The only difference from the definition of $\mathcal{W}_{d,g}^r$ is that here exact equality is required.

This sequence consists of $(r + 1)$ distinct integers; it is usually denoted $a^L(p) = (a_0^L(p), a_1^L(p), \dots, a_r^L(p))$ where $a_0^L(p) < a_1^L(p) < \dots < a_r^L(p)$. Equivalent to the vanishing sequence is the *ramification sequence* $\alpha^L(p)$, given by $\alpha_i^L(p) = a_i^L(p) - i$. Most authors work with the ramification sequence rather than the vanishing sequence; we will work almost entirely with the vanishing sequence since it is slightly more notationally convenient for our purposes.

Let $\tilde{\mathcal{G}}_{d,g}^r(a) \subseteq \mathcal{G}_{d,g}^r \times_{\mathcal{M}_g} \mathcal{M}_{g,1}$ denote the space of triples (C, L, p) such that the vanishing sequence of L at p is precisely a . Let $\mathcal{G}_{d,g}^r(a)$ denote the space of such triples such that the vanishing sequence of L at p is at least⁴ a . More generally, $\tilde{\mathcal{G}}_{d,g}^r(a^1, a^2, \dots, a^s) \subseteq \mathcal{G}_{d,g}^r \times_{\mathcal{M}_g} \mathcal{M}_{g,s}$ denotes the space of tuples (C, L, p_1, \dots, p_s) with vanishing sequence a^i at p_i .

The theory of limit linear series works best for curves of compact type. A nodal curve X is called *compact type* if its dual graph (that is, the graph whose vertices are the components of X and whose edges correspond to the nodes) has no cycles (equivalently, the Jacobian of X is compact). Recently, Amini and Baker gave a definition of limit linear series for arbitrary nodal curves, but there does not yet exist a moduli space for these more general limit linear series. We will use the original definitions of Eisenbud and Harris.

Definition 4.1. Let X be a curve of compact type. A *refined limit linear series* L of degree d and rank r (or *limit* \mathfrak{g}_d^r) on X consists of a $\mathfrak{g}_d^r L^C$ on each connected component C of X (called the C -aspect of L), such that for each node $p \in X$ joining components C_1 and C_2 , the following compatibility condition holds.

$$a_i^{L^{C_1}}(p) + a_{r-i}^{L^{C_2}}(p) = d \text{ for } i = 0, 1, 2, \dots, r$$

The *vanishing sequence* $a^L(p)$ of a limit series at a smooth point p is the vanishing sequence of the C -aspect of L , where $p \in C$.

Eisenbud and Harris also define *coarse* limit series to be a collection of C -aspects such that the compatibility condition holds as an inequality. We will not need to consider coarse limits in this paper. Note that Osserman [12] gave a different definition of limit linear series that is more suitable for the construction of a global moduli scheme. His definition is equivalent to the Eisenbud-Harris definition in the special case of refined limit series. Eisenbud and Harris do not construct a global moduli space of limit \mathfrak{g}_d^r s over all of \mathcal{M}_g , but instead construct a local moduli space. More precisely, they construct a moduli space of (refined) limit linear series over a Kuranishi family of any curve of compact type. In either formalism, the existence of a suitable moduli space, plus a dimension bound on it coming from Schubert conditions, implies the following “regeneration theorem.” To state it first requires one more definition.

Definition 4.2. A marked curve (X, p_1, \dots, p_s) with a linear series L of degree d and rank r is called *dimensionally proper* if the local dimension of $\mathcal{G}_{d,g}^r(a^L(p_1), \dots, a^L(p_s))$ is exactly

$$\dim \mathcal{M}_{g,s} + \rho - \sum_{i=1}^s \sum_{j=0}^r (a_j^L(p_i) - j).$$

Theorem 4.3 (Corollary 3.7 of [6]). *Let L be a limit \mathfrak{g}_d^r on a curve X of compact type, and $p_1, \dots, p_s \in X$ are smooth points. Suppose that each component C of X is dimensionally proper with respect to all the points of C that are nodes in X and all the marked points p_i that lie on C . Then there exists a smooth marked curve (X', p'_1, \dots, p'_s) with a dimensionally proper $\mathfrak{g}_d^r L'$ such that $a^{L'}(p'_j) = a^L(p_j)$ for all j . This marked curve and linear series lies in a one-parameter family whose limit is the marked curve (X, p_1, \dots, p_s) with limit linear series L .*

Definition 4.4. A marked curve (X, p_1, \dots, p_s) of compact type with a refined limit L satisfying the hypotheses of theorem 4.3 will also be called *dimensionally proper*.

The following lemma reinterprets the data of a twisted Weierstrass point in a manner that makes the theory of limit linear series applicable.

⁴Whenever we say that a sequence a is “at least” another sequence a' , we mean that $a_i \geq a'_i$ for each i .

Lemma 4.5. *Let $r \geq g - 1$ be an integer. Then $\tilde{W}_g(P) \cong \tilde{\mathcal{G}}_{r+g,g}^r(a)$, where $a = (a_0, a_1, \dots, a_r)$ is the sequence given by $a_i = i + P_{r-i}$, via the maps $(C, \mathcal{L}, p) \mapsto (C, |\mathcal{L}((r+g)p)|, p)$ and $(C, (\mathcal{L}, H^0(\mathcal{L})), p) \mapsto (C, \mathcal{L}(-(r+g)p), p)$.*

Proof. Since $r+g \geq 2g-1$, $|\mathcal{L}((r+g)p)|$ is indeed a \mathfrak{g}_{r+g}^r ; unraveling definitions shows that the vanishing sequence at p is a . So this is a well-defined map to $\tilde{\mathcal{G}}_{r+g,g}^r(a)$. In reverse, every \mathfrak{g}_{r+g}^r is necessarily complete, hence of the form $|\mathcal{L}|$ for some \mathcal{L} ; then $(C, \mathcal{L}(-(r+g)p), p)$ indeed lies in $\tilde{W}_g(P)$ by the same calculation. \square

Therefore, we have the following notion of a *limit twisted Weierstrass point*: a curve X of compact type, with marked smooth point p and refined limit $\mathfrak{g}_{r+g}^r L$ (where $r \geq g-1$) and the vanishing sequence a as described above. Constructing such object, and proving that they are dimensionally proper (in the sense of definition 4.4) will suffice to construct dimensionally proper twisted Weierstrass points (on smooth curves).

5 Elliptic bridges and displacement

The object of this section is to demonstrate how dimensionally proper twisted Weierstrass points on genus g curves give rise to dimensionally proper twisted Weierstrass points on curves of genus $g+1$, with slightly modified partitions. The construction proceeds by adjoining an elliptic curve to the genus g curve, and smoothing the resulting nodal curve. The basic technical tool is the regeneration theorem for limit linear series, as introduced by Eisenbud and Harris [6] (see [11] for a readable expository account and [12] for a more recent perspective that is more applicable in characteristic p).

The following lemma is a slight restatement of proposition 5.2 from [4]. It is the basic tool in our inductive constructions.

Lemma 5.1. *Fix integers r, d and two sequences $b = (b_0, b_1, \dots, b_r)$ and $c = (c_0, c_1, \dots, c_r)$ such that*

$$b_i + c_{r-i} = d - 1$$

for each index i . Then for any genus 1 curve E with distinct points p, q and degree d line bundle \mathcal{L} , there exists a unique linear series $L = (\mathcal{L}, V)$ on E such that for all i the following inequalities hold.

$$\begin{aligned} a_i^L(p) &\geq b_i \\ a_i^L(q) &\geq c_i \end{aligned}$$

Proof. For all pairs of indices (i, j) with $i+j < d$, define the following vector space of sections of \mathcal{L} .

$$W_{i,j} = \text{im} (H^0(\mathcal{L}(-ip - jq)) \hookrightarrow H^0(\mathcal{L}))$$

That is, $W_{i,j}$ consists of those sections vanishing to order at least i at p and at least order j at q . $W_{i,j}$ has dimension $d - i - j$, by Riemann-Roch.

Separate the indices $\{0, 1, 2, \dots, r\}$ into the longest intervals $I_k = \{u_k, u_k+1, \dots, v_k\}$, such that $b_{u_k}, b_{u_k+1}, \dots, b_{v_k}$ are consecutive integers. Let m be the number of these intervals, so that $\{0, 1, \dots, r\}$ is a disjoint union of I_1, I_2, \dots, I_m . Then $u_1 = 0, v_m = r$, and $v_k + 1 = u_{k+1}$. Define, for each $k \in \{1, 2, \dots, m\}$, $V_k := W_{b_{u_k}, c_{r-v_k}}$. Observe that the dimension of V_k is $d - b_{u_k} - b_{r-v_k} = d - b_{u_k} - (d-1) + b_{v_k} = 1 + b_{v_k} - b_{u_k} = 1 + v_k - u_k = |I_k|$.

Let V be the sum of all the spaces V_k . We claim that V satisfies the conditions of the lemma, and that it is the unique such vector space of sections. First, we verify that V satisfies the conditions of the lemma. By the Riemann-Roch formula, each vector space V_k has the following orders of vanishing at p : $\{b_{u_k}, b_{u_k+1}, \dots, b_{v_k-1}, b'_{v_k}\}$, where

$$b'_{v_k} = \begin{cases} b_{v_k} + 1 & \text{if } \mathcal{L} \cong \mathcal{O}_E((b_{v_k} + 1)p + (c_{r-v_k})q) \\ v_k & \text{otherwise.} \end{cases}$$

In all cases, $b'_{v_k} < b_{u_k+1}$, so the sections of any two different spaces V_k have disjoint sets of orders of vanishing at p . It follows that the orders of vanishing at p of sections in V is the disjoint union

$$\bigcup_{k=1}^m \{b_{u_k}, b_{u_k+1}, \dots, b_{v_k-1}, b'_{v_k}\}.$$

In particular, the dimension of V is $\sum_{k=1}^m |I_k| = r + 1$, and its vanishing sequence at p is at least $b_{u_1}, b_{u_1+1}, \dots, b_{v_1}, b_{u_2}, \dots, b_{v_2}, \dots, b_{v_m}$, which is identical to b_0, b_1, \dots, b_r . Symmetric reasoning shows that the orders of vanishing of V at q are at least c_0, c_1, \dots, c_r . So V satisfies the conditions of the lemma.

Now suppose that V' satisfies the conditions of the lemma. Then the sections of V' vanishing to order at least a_{u_k} have codimension at most u_k , and those vanishing to order at least b_{r-v_k} at q have codimension at most $r - v_k$, hence $V' \cap V_k$ has codimension at most $r + (u_k - v_k)$ and thus dimension at least $1 + v_k - u_k = |I_k|$. Therefore this intersection must be all of V_k . Thus $V' \supseteq V$, and $\dim V' = \dim V$, so in fact $V' = V$. So V is the unique such vector space of sections. \square

In fact, examining the end of the proof of lemma 5.1, we have actually proved the following.

Lemma 5.2. *Let $L = (\mathcal{L}, V)$ be a linear series as described in lemma 5.1. Then the actual orders of vanishing of L are as follows:*

$$\begin{aligned} a_i^L(p) &= \begin{cases} b_i + 1 & \text{if } (b_i + 1) \in \Lambda \text{ and } b_{i+1} > b_i + 1 \\ b_i & \text{otherwise} \end{cases} \\ a_i^L(q) &= \begin{cases} c_i + 1 & \text{if } (c_i + 1) \in (d - \Lambda) \text{ and } c_{i+1} > c_i + 1 \\ c_i & \text{otherwise} \end{cases} \end{aligned}$$

where Λ is the arithmetic progression $\{n : \mathcal{L} \cong \mathcal{O}_E(np + (d - n)q)\}$. \square

For notational convenience, we make the following definition. In the following definition and the remainder of this paper, an *arithmetic progression* will be a proper subset Λ of the integers such that the set of differences of elements of Λ is closed under addition. In particular, Λ may be empty or have only a single element, but it may not be all of \mathbf{Z} .

Definition 5.3. Let $a = (a_0, a_1, a_2, \dots, a_r)$ be a strictly increasing sequence of integers, and let Λ be an arithmetic progression (as defined above). Define the *upward displacement* a_Λ^+ and *downward displacement* a_Λ^- of a with respect to Λ as follows.

$$\begin{aligned} (a_\Lambda^+)_i &= \begin{cases} a_i + 1 & \text{if } a_i + 1 \in \Lambda \text{ and } a_{i+1} > a_i + 1 \\ a_i & \text{otherwise} \end{cases} \\ (a_\Lambda^-)_i &= \begin{cases} a_i - 1 & \text{if } a_i \in \Lambda \text{ and } a_{i-1} < a_i - 1 \\ a_i & \text{otherwise} \end{cases} \end{aligned}$$

In these expressions i is an index in $\{0, 1, \dots, r\}$ and for notational convenience $a_{-1} = -\infty$ and $a_{r+1} = \infty$ (when these appear on the right side). This definition is interpreted visually, using partition notation, in figure 2.

Informally, the upward displacement “attracts” the sequence upward to the progression Λ , while the downward displacement “repels” the sequence downward away from Λ . Another interpretation is that displacement forgets, for each pair $\{\lambda - 1, \lambda\}$ (where $\lambda \in \Lambda$) which of these two numbers is in the sequence, remembering only how many (0, 1, or 2) are present.

The following lemma reformulates the previous two lemmas in the language of limit linear series.

Lemma 5.4. *Let C be a smooth curve, $p_1 \in C$ a point, E a genus 1 curve, and p_2, q two distinct points on E . Let X be the nodal curve obtained by attaching C and E at p_1 and p_2 . Let $L^C = (\mathcal{L}^C, V^C)$ be a \mathfrak{g}_d^r on C , and \mathcal{L}^E a degree $(d+1)$ line bundle on E . Then there exists a unique limit \mathfrak{g}_{d+1}^r L on X with the following properties.*

1. *The C -aspect of L is $L^C + p_1$ (that is, L^C with a base point added at p_1).*
2. *The E -aspect of L has line bundle \mathcal{L}^E .*
3. *For all $i \in \{0, 1, \dots, r\}$, $a_i^L(q) \geq a_i^{L^C}(p_1)$.*

Let $\Lambda = \{n : \mathcal{L}^E \cong \mathcal{O}_E(nq + (d+1-n)p_2)\}$; then the vanishing sequence of L is precisely $a^L(q) = (a^{L^C}(p_1))_\Lambda^+$, and L is a refined limit linear series if and only if $(a^{L^C}(p_1))_\Lambda^- = a^{L^C}(p_1)$.

Proof. For $i \in \{0, 1, \dots, r\}$, let $b_i = d - a_{r-i}^{L^C}(p_1)$ and let $c_i = a_i^{L^C}(p_1)$. Then of course $b_i + c_{r-i} = (d+1) - 1$ for each i , so a suitable E -aspect for L exists and is unique by lemma 5.1. Lemma 5.2 shows that vanishing sequence of this E -aspect (and therefore of L) is c_Λ^+ as claimed. The vanishing sequence of the E -aspect at p_2 is $b_{(d+1-\Lambda)}^+$, and L is refined if and only if this is equal to b . But observe that since $b_i = d - c_{r-i}$, this is equivalent to $c_\Lambda^- = c$, as claimed. \square

By allowing the curve X and limit linear series L to vary, this construction on two-component curves gives the following result on dimensionally proper linear series with specified ramification.

Proposition 5.5. *Suppose that $a = (a_0, a_1, \dots, a_r)$ is a strictly increasing sequence of nonnegative integers, and Λ is an arithmetic progression (as defined above) such that $a_\Lambda^- = a$ and a_Λ^+ differs from a in at most two places. If $\tilde{\mathcal{G}}_{d,g}^r(a)$ has a dimensionally proper point, belonging to a component mapping to $\mathcal{M}_{g,1}$ with general fiber dimension d , then $\tilde{\mathcal{G}}_{d+1,g+1}^r(a_\Lambda^+)$ has a dimensionally proper point, belonging to a connected component mapping to $\mathcal{M}_{g,1}$ with general fiber dimension at most $d+1$ (if $a = a_\Lambda^+$) or at most d (if $a \neq a_\Lambda^+$).*

Proof. Assume without loss of generality that Λ is as small as possible. This means that if a_Λ^+ differs from a in two places, then Λ is the progression generated by those two values $(a_\Lambda^+)_i$ that are greater than a_i ; if a_Λ^+ differs from a in one place, then Λ is a single element; and if $a_\Lambda^+ = a$, then Λ is empty.

Let (C, L^C, p_1) be a dimensionally proper point of $\tilde{\mathcal{G}}_{d,g}^r(a)$. Let (E, \mathcal{L}, p_2, q) be a twice-pointed elliptic curve, chosen in the following way.

- If Λ is infinite, say $\{n : n \equiv m \pmod{d}\}$ for some m and d , then let $\mathcal{L} = \mathcal{O}_E(mq + (d+1-m)p_2)$ and select p_2, q so that $(p_2 - q)$ is a d -torsion point on $\text{Pic}^0(E)$.
- If Λ has a single element m , then let $\mathcal{L} = \mathcal{O}_E(mq + (d+1-m)p_2)$ and choose p_2, q so that $(p_2 - q)$ is not torsion.
- If Λ is empty, then choose \mathcal{L} distinct from all line bundles $\mathcal{O}_E(mq + (d+1-m)p_2)$ and choose p_2, q arbitrarily.

Let X be the nodal curve described in lemma 5.4, L the limit \mathfrak{g}_{d+1}^r on X described in that lemma, and L^E its E -aspect. Since $a_\Lambda^- = a$, this series is refined. We shall show that (X, L, q) is dimensionally proper in the sense described in section limits. By assumption, the C -aspect $L^C + p_1$, with the marked point p_1 , is dimensionally proper. So it suffices to prove that (E, L^E, p_2, q) is dimensionally proper. Let δ be the number of places where a_Λ^+ differs from a . An elementary calculation shows that this is equivalent to showing that the local dimension of $\mathcal{G}_{d+1,1}^r(a^{L^E}(p_2), a^{L^E}(q))$ at (E, L^E, p_2, q) is $3 - \delta$. Now, the map $f : \mathcal{G}_{d+1,1}^r(a^{L^E}(p_2), a^{L^E}(q)) \rightarrow \text{Pic}_{1,2}^{d+1}$ (that is, to the moduli space of twice-marked genus 1 smooth curves with a chosen degree $(d+1)$ line bundle) is set-theoretically injective by lemma 5.1. By lemma 5.2, the image of f consists of all $(E', \mathcal{L}', p'_2, q')$ such that the arithmetic progression $\Lambda' = \{n : \mathcal{L}' \cong \mathcal{O}_{E'}(np'_2 + (d+1-n)p'_2)\}$ contains Λ . By a little casework, the dimension of the image is $3 - \delta$. It follows that (E, L^E, p_2, q) is dimensionally proper, and therefore so

is (X, L, q) . By theorem 4.3, $\tilde{\mathcal{G}}_{d+1, g+1}^r(a_\Lambda^+)$ has a dimensionally proper point. The bound on the dimension of fibers over $\mathcal{M}_{g,1}$ follows by considering the semicontinuity of fiber dimension for the map from the space of limit linear series (on a 1-parameter family degenerating to (X, p)) over $\bar{\mathcal{M}}_{g,1}$. \square

6 The displacement difficulty of a partition

As before, we will use the following convention: an *arithmetic progression* will mean a proper subset $\Lambda \subset \mathbf{Z}$ such that $\Lambda - \Lambda$ is closed under addition. In particular, Λ may be empty or have a single element, but it cannot be all of \mathbf{Z} . Also, we adopt the following notational conventions: the partition elements are $P_0 \geq P_1 \geq \dots \geq P_n$, and P_k is defined to be 0 for $k > n$ and ∞ for $k < 0$.

Definition 6.1. Let P be a partition and Λ an arithmetic progression. Then define the *upward displacement* P_Λ^+ and *downward displacement* P_Λ^- of P with respect to Λ as follows.

$$\begin{aligned} (P_\Lambda^+)_i &= \begin{cases} P_i + 1 & \text{if } (P_i - i) \in \Lambda \text{ and } P_{i-1} > P_i \\ P_i & \text{otherwise} \end{cases} \\ (P_\Lambda^-)_i &= \begin{cases} P_i - 1 & \text{if } (P_i - i - 1) \in \Lambda \text{ and } P_{i+1} < P_i \\ P_i & \text{otherwise} \end{cases} \end{aligned}$$

This definition is much easier to understand visually; it is illustrated in figure 2. Here the partition P is represented by its Young diagram, and the arithmetic progression Λ is represented by an evenly spaced sequence of diagonal lines. Then the two displacements are obtained by finding all places where the line of Λ meet the corners of P , and either “turning the corners out” (in the case of P_Λ^+) or “turning the corners in” (in the case of P_Λ^-).

Observe that if P' is any other partition such that $P_\Lambda^- \leq P' \leq P_\Lambda^+$, then the upward and downward displacements of P' are the same as those of P (with respect to Λ). So displacement can be regarded as a sort of projection to the nearest partition that is stable with respect to the given arithmetic progression.

Call two partitions P_1, P_2 *linked* if there is an arithmetic progression Λ (proper but possibly empty or singleton) such that P_2 is the upward displacement of P_1 and P_1 is the downward displacement of P_2 . Note that this implies that P_1 is its own downward displacement and P_2 is its own upward displacement. Say that P_1 and P_2 are *k-linked* if they are linked and $|P_2| - |P_1| = k$.

It is easy to verify that if P_1, P_2 are any two partitions with $P_1 \leq P_2$, then P_1 can be connected to P_2 by a sequence of 1-linked partitions. Indeed, the arithmetic progressions can be taken to be singletons.

As we saw in the previous section, we are particularly interested in 2-linked partitions. More specifically, we are interested in partitions that can be joined by a path of 1-linked and 2-linked pairs, using as few 1-linked pairs as possible. Therefore make the following definition.

Definition 6.2. Call a sequence of partitions of increasing sum *valid* if any two adjacent partitions in the sequence are 1-linked or 2-linked. Define the *difficulty* $\delta(P)$ of a partition P to be the fewest number of 1-linked adjacent pairs in a valid sequence from the empty partition to P .

With this definition, we can now state the following lemma, which relates difficulties of partitions to dimensionally proper twisted Weierstrass points.

Lemma 6.3. Let P be any partition and Λ an arithmetic progression (proper and possibly empty or singleton). If $|P_\Lambda^+| - |P_\Lambda^-| \leq 2$ and $\tilde{\mathcal{W}}_g(P_\Lambda^-)$ has a dimensionally proper point lying in a fiber over $\mathcal{M}_{g,1}$ of dimension d , then $\tilde{\mathcal{W}}_g(P_\Lambda^+)$ has a dimensionally proper point lying in a fiber of dimension at most $(d+1)$, and at most d if $P_\Lambda^+ \neq P_\Lambda^-$.

Proof. Without loss of generality, let $P = P_\Lambda^-$. Let $(C, \mathcal{L}, p) \in \tilde{\mathcal{W}}_g(P)$ be a dimensionally proper point. By lemma 4.5, this can also be regarded as a dimensionally proper point of $\tilde{\mathcal{G}}_{r+g, g}^r(a)$, where $r = g$, for $a_i = i + P_{r-i}$. Let $\Lambda' = \Lambda + (r+1)$. Then it follows that, again by lemma 4.5, $\tilde{\mathcal{W}}_g(P_\Lambda^+) \cong \tilde{\mathcal{G}}_{r+g, g}^r(a_\Lambda^+)$. Since

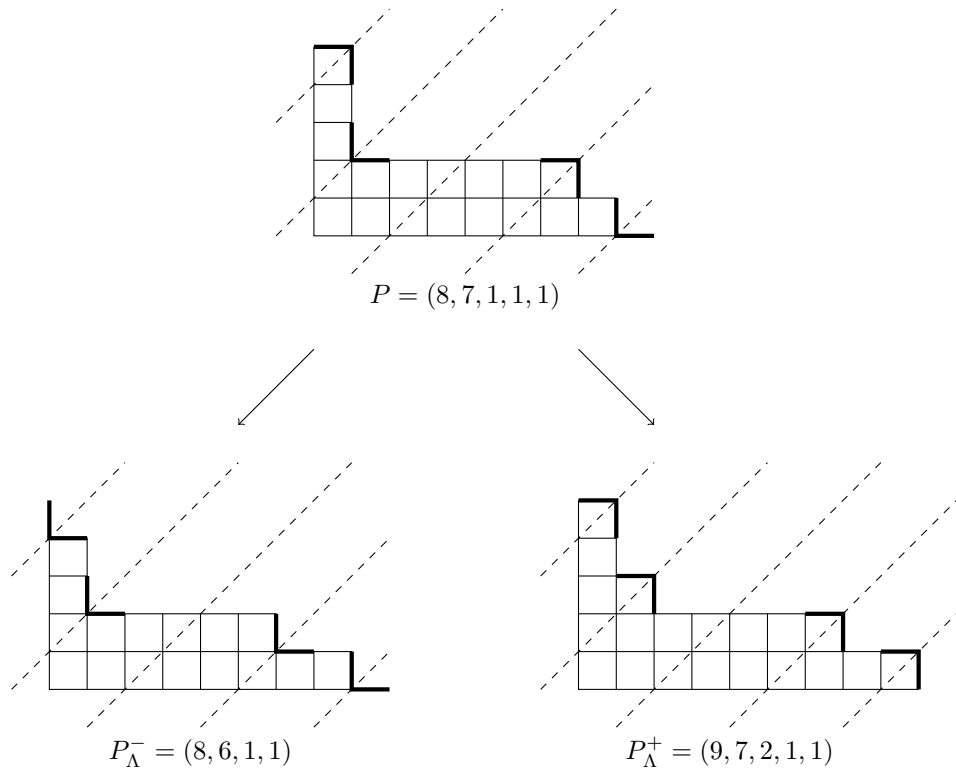


Figure 2: An example illustrating the definition of displacement. Here $\Lambda = \{2 \bmod 3\}$.

a differs from a_Λ^+ in at most 2 places, proposition 5.5 implies that $\tilde{\mathcal{W}}_g(P_\Lambda^+)$ has a dimensionally proper point, lying in a fiber over $\mathcal{M}_{g,1}$ of dimension at most $d+1$ (at most d if $P_\Lambda^+ \neq P_\Lambda^-$). \square

Corollary 6.4. *Let P be any partition. Then for all $g \geq \frac{1}{2}(|P| + \delta(P))$, $\tilde{\mathcal{W}}_g(P)$ has a dimensionally proper point, lying in a fiber over $\mathcal{M}_{g,1}$ of dimension at most $\max(0, g - |P|)$.*

To prove theorem 1.1, we are interesting in bounding the difficulty of “box-shaped” partitions, i.e. partitions of the form (a^b) . The table below shows some experimental data about the difficulties of these partitions for various values of a and b .

	2	3	4	5	6	7	8	9	10	11	12
2	2	4	4	6	6	6	6	8	8	10	10
3	4	5	6	7	6	7	8	7	6	7	6
4	4	6	4	6	6	8	4	6	6	6	4
5	6	7	6	7	6	5	6	5	4	5	6
6	6	6	6	6	6	4	4	4	4	4	4
7	6	7	8	5	4	7	4	5	6	5	6
8	6	8	4	6	4	4	4	4	4	6	4
9	8	7	6	5	4	5	4	5	4	5	4
10	8	6	6	4	4	6	4	4	4	4	6
11	10	7	6	5	4	5	6	5	4	7	4

On the basis of these experimental data, we make the following conjecture.

Conjecture 6.5. There is a constant C such that for all positive integers $a, b \geq 3$, $\delta((a^b)) \leq C$.

The assumption $a, b \geq 3$ is harmless, since if $a = 2$ then the corresponding twisted Weierstrass points detect \mathfrak{g}_d^1 s, whose moduli are well-understood.

Remark 6.6. It is apparent from the definition that $\delta(P) = \delta(P^*)$ where P^* is the conjugate partition. This is not surprising, in light of the duality $\tilde{\mathcal{W}}_g(P) \cong \tilde{\mathcal{W}}_g(P^*)$.

Remark 6.7. Corollary 6.4 is equivalent, in the special case of box-shaped partitions, to saying that if $P = ((r+1)^{g-d+r})$, and $\rho \geq -g + \delta(P)$, then $\tilde{\mathcal{W}}_{d,g}^r$ has a dimensionally proper point. So if conjecture 6.5 is true, it would should that that the “phase transition” from dimensionally proper to improper occurs very close to $\rho = -g$, as we expect. We suspect that conjecture 6.5 is tractable, but do not yet have a proof. In section 8, we prove a somewhat weaker bound, linear in a and b , that is sufficient to give theorem 1.1.

7 Primitive Weierstrass points

As an example and first application of the techniques described above, we will prove one of the main results of [4] on the existence of dimensionally proper Weierstrass points.

To state the result requires a bit of terminology. A subset $S \subseteq \mathbb{Z}_{\geq 0}$ that contains 0 and is closed under addition is called a *numerical semigroup*. The size of the complement is called the *genus*. The sum of the elements of the complement, minus $\binom{g+1}{2}$, is called the *weight*. A semigroup is called *primitive* if twice the smallest positive element is greater than all the gaps; this is equivalent to saying that for all sets $S' \geq S$ whose complement is size g , S' is also a semigroup. Let $\mathcal{C}_S \subseteq \mathcal{M}_{g,1}$ be the locus of Weierstrass points with semigroup S ; a point of \mathcal{C}_S is *dimensionally proper* if the local codimension of \mathcal{C}_S in $\mathcal{M}_{g,1}$ is equal to the weight of S .

Theorem 7.1 (Eisenbud and Harris). *Let S be a primitive numerical semigroup of weight at most $g - 2$. Then \mathcal{C}_S has dimensionally proper points⁵.*

In fact the primitivity assumption is not an artifact of the proof, but is a crucial assumption. Our method also gives an easy proof of the following.

⁵These theorem was improved by Komeda [10], who replaced $g - 2$ by $g - 1$; see remark 7.4.

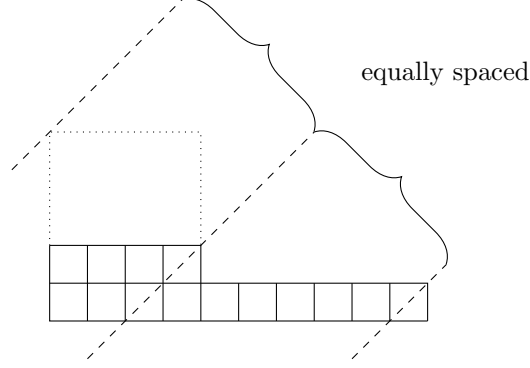


Figure 3: Visual representation of the primitivity condition. Draw diagonals through the upper-right corners of the first two rows of the Young diagram, and also a third diagonal, equally spaced from the second. Then the rest of the Young diagram must fit below this third diagonal.

Theorem 7.2. *If S is a non-primitive semigroup, then the moduli space \mathcal{C}_S of pointed curves with Weierstrass semigroup S has no dimensionally proper points.*

Although this fact is not explicitly proved in [4], there are other ways to establish it; we give another argument in [13], based on the notion of the effective weight of a semigroup.

First we re-express theorem 7.1 using the notation of this paper. Notice that $\mathcal{C}_S \cong \tilde{\mathcal{W}}_g(P)$, where P is the partition given by $P_n = (g + n) - s_n$ (where $0 = s_0 < s_1 < s_2 < \dots$ are the elements of S). That S is a primitive semigroup is equivalent to saying that $2(g + 1 - P_1) \geq (g + P_0^*)$ (where P^* is the dual partition), which is equivalent, using the fact that $P_0 = g$, to $P_0 - P_0^* \geq 2P_1 - 2$. Thus theorem 7.1 follows from the following, by lemma 6.4.

Lemma 7.3. *Let P be any partition such that $P_0 - P_0^* \geq 2P_1 - 2$ and $|P| \leq 2P_0 - 2$. Then $\delta(P) = 2P_0 - |P|$.*

Proof. First, notice that any valid sequence of partitions ending in P must have at least P_0 steps, since P_0 can increase by at most 1 at each step. This means that $\frac{1}{2}(|P| + \delta(P)) \geq P_0$, i.e. $\delta(P) \geq 2P_0 - |P|$. So it suffices to show the opposite inequality.

The opposite inequality follows by induction on P . As the base case, consider the case $P_1 = 0$. Then $|P| = P_0 = \delta(P)$, so the result follows. So assume that $P_1 > 0$. Let $k \geq 1$ be the largest integer such that $P_k = P_1$. Then let Λ be the arithmetic progression generated by $P_0 - 1$ and $P_k - k - 1$. The corresponding diagonal lines meet the Young diagram of P at only two corners, both outward, at the ends of rows 0 and k (see figure 3). Thus $P_\Lambda^+ = P$ and P_Λ^- differs in exactly two places from P : P_0 and P_k are both decreased by 1. Now, it is immediate that $|P_\Lambda^-| \leq 2(P_\Lambda^-)_0 - 2$. It remains to show that $(P_\Lambda^-)_0 - (P_\Lambda^-)_0^* \geq 2(P_\Lambda^-)_1 - 2$. Since P_0 decreased by 1 under the displacement, the only way that this inequality could fail is if P_0^* is unchanged, P_1 is unchanged, and the inequality was sharp before, i.e. $P_0 - P_0^* = 2P_1 - 2$. This would mean that $P_1 = P_2$ and the Young diagram meets the third diagonal in figure 3; see figure 4. But in this case, we would have $|P| \geq P_0 + 2P_1 + (P_0^* - 3) = 2P_0 - 1$, which contradicts the assumption that $|P| \leq 2P_0 - 2$. Hence P_Λ^- satisfies the hypotheses of the lemma. Also, it is clear that $2P_0 - |P|$ is unchanged and $\delta(P_\Lambda^-) \geq \delta(P)$, so the desired inequality follows by induction. completing the induction. \square

Remark 7.4. Notice that the proof above very nearly shows the existence of all dimensionally proper Weierstrass points of weight less than g (rather than $g - 1$). If we attempt to prove this slightly stronger statement by an identical induction, we see that the inductive step fails only when $P_1 = P_2$, $P_3 = 1$, and $P_0^* = P_0 - 2P_1 + 2$ (i.e. the area enclosed by the dashed line in figure 4 is empty). A different displacement works in this case, namely by turning in the first and last outward corner, unless $P_3 = 0$. So the only partitions that cannot be treated this way are $P = ((2m - 1) \ m \ m)$. Komeda [10] proved, by a different

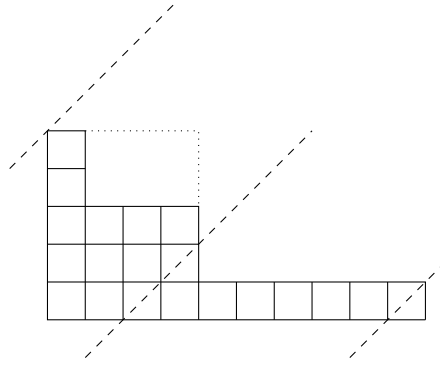


Figure 4: The only situation where primitivity may fail after displacement, in the proof of theorem 7.1

method, that dimensionally proper Weierstrass points corresponding to these partitions exist. So by adding these partitions as an additional base case, Komeda extended theorem 7.1 to all primitive semigroups of weight less than g .

Using the same technique of displacement along elliptic curves, we can also prove the non-existence of dimensionally proper Weierstrass points. This result is substantially generalized, by a different method, in [13]. We include this proof because it demonstrated the capability of the technique of displacement to disprove the existence of dimensionally proper points as well.

Proof of theorem 7.2. Suppose for the sake of contradiction that $S = \{0, s_1, s_2, \dots\}$ is a non-primitive semi-group such that \mathcal{C}_S has a dimensionally proper point. Let P be the corresponding partition, so that $P_0 = g$ and $\tilde{\mathcal{W}}_g(P)$ has a dimensionally proper point. Define P^k to be the partition given by $P_0^k = P_0 + k$ and $P_i^k = P_i$ otherwise. By displacing repeatedly along singleton arithmetic progressions, it follows that $\tilde{\mathcal{W}}_{g+k}(P^k)$ has a dimensionally proper point (for each k). This corresponds to a dimensionally proper Weierstrass point in \mathcal{C}_{S^k} , where $S^k = \{0, s_1 + k, s_2 + k, \dots\}$. Since S is not primitive, there exists a positive integer $f > 2s_1$ such that $f \notin S$. Let $k = f - 2s_1$. Then $s_1 + k \in S^k$, but $2(s_1 + k) = f + k \notin S^k$. This is a contradiction; so \mathcal{C}_S cannot have any dimensionally proper points. \square

8 Special linear series

In order to prove the existence of a reasonably large class of dimensionally proper linear series, it suffices, by lemma 6.4 to bound the displacement difficulty of box-shaped partitions. We shall prove the following bound, which is likely to be very far from optimal, but is strong enough to give theorem 1.1.

Lemma 8.1. *Let P be the partition (a^b) , i.e. the partition of the number ab into b equal parts, where $a, b \geq 2$. Then $\delta(P) \leq a + 3b - 5$.*

The proof appears at the end of this section. This lemma, together with lemma 6.4 is sufficient to complete the proof of the main theorem.

Proof of theorem 1.1. Suppose that g, d, r are integers such that $r \geq 1$, $g - d + r \geq 2$, and $0 > \rho \geq -\frac{r}{r+2}g + 3r - 3$. Let $a = g - d + r$, $b = r + 1$, and $P = (a^b)$. Note that $a, b \geq 2$. By lemma 3.5, $\tilde{\mathcal{W}}_{d,g}^r$ has a dimensionally proper point if and only if $\tilde{\mathcal{W}}_g(P)$ has a dimensionally proper point. By lemmas 6.4 and 8.1, it suffices to show the $g \geq \frac{1}{2}(ab + a + 3b - 5)$. We may assume that $a, b \geq 3$ since the case $r = 1$ is well-understood (by the duality mentioned in remark 3.6, the roles of a and b can be interchanged, so either can be taken to be $r + 1$).

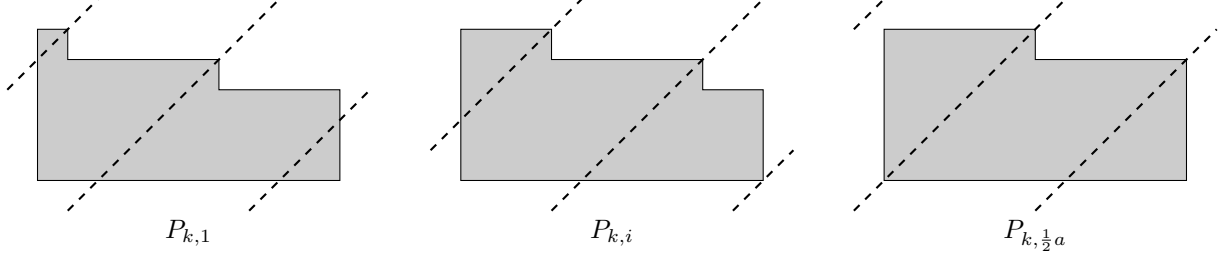


Figure 5: The intermediate partitions $P_{k,i}$ used in the proof of lemma 8.1, together with the progressions $\Lambda_{k,i}$. The partition is its own upward displacement for all values of i except possibly one (shown in the middle).

Now, $\rho \geq -\frac{r}{r+2}g + 3r - 3$ is equivalent to $g - ab \geq \frac{b-1}{b+1}g + 3b - 6$, i.e. $\frac{2b}{b+1}g \geq ab + 3b - 6$. This is equivalent to $2g \geq (a+3)(b+1) - \frac{6(b+1)}{b} = ab + a + 3b - 3 - \frac{6}{b}$. Since $b \geq 3$, this implies that $2g \geq ab + a + 3b - 5$, and the theorem follows. \square

Proof of lemma 8.1. The proof will be by explicit construction of a sequence of partitions. First consider the case where a is **even**.

Define the following intermediate partitions: $P_{k,i} = (a^k (i + \frac{1}{2}a) i)$ (see figure 5), for $k \geq 0$ and $i \in \{0, 1, \dots, \frac{1}{2}a\}$.

Let $\Lambda_{k,i}$ denote the arithmetic progression generated by the two diagonals shown in figure 5. That is, $\Lambda_{k,i} = \{n : n \equiv i - k - 2 \pmod{(\frac{1}{2}a + 1)}\}$. Observe that if $1 \leq i \leq \frac{1}{2}a$, then $\Lambda_{k,i}$ does not meet the other outward-facing corner of the Young diagram, so it follows that

$$(P_{k,i})_{\Lambda_{k,i}}^- = P_{k,i-1} \text{ when } i > 0.$$

Now consider the upward displacement. The only inward-turned corner that $\Lambda_{k,i}$ can meet is the one at the end of the first row of the Young diagram; this corresponds to the value $P_0 - 0 = a$. From this we can conclude that

$$(P_{k,i})_{\Lambda_{k,i}}^+ = P_{k,i} \text{ unless } k > 0 \text{ and } a \equiv i - k - 2 \pmod{(\frac{1}{2}a + 1)}.$$

For a fixed positive value of k , there is at most one value $i \in \{1, 2, \dots, \frac{1}{2}a\}$ such that congruence above holds. Therefore the sequence of partitions

$$P_{k,0} < P_{k,1} < \dots < P_{k,\frac{1}{2}a}$$

is nearly a valid sequence of partitions; at most one adjacent pair is invalid. By inserting an intermediate partition at that place (if necessary), we obtain a valid sequence of partitions with at most two steps increasing the sum by only 1. Therefore $\delta(P_{k,\frac{1}{2}a}) \leq 2 + \delta(P_{k,0})$. For $k = 0$, the original sequence is valid, so $\delta(P_{0,\frac{1}{2}a}) \leq \delta(P_{0,0})$.

Since $P_{k,\frac{1}{2}a} = P_{k+1,0}$, it follows from this analysis that

$$\delta(P_{b-1,0}) \leq 2(b-2) + \delta(P_{0,0}).$$

Now, $P_{b-1,0} \leq (a^b)$ with $|(a^b)| - |P_{b-1,0}| = \frac{1}{2}a$ and $|P_{0,0}| = \frac{1}{2}a$. From this it follows (by a sequence of displacements along singleton progressions) that

$$\delta((a^b)) \leq a + 2b - 4 \text{ when } a \text{ is even.}$$

Now, if a is odd, then $\delta(((a-1)^b)) \leq a+2b-5$, and $((a-1)^b)$ can be linked to (a^b) by a length b sequence of length b . Therefore

$$\delta((a^b)) \leq a+3b-5 \text{ when } a \text{ is odd.}$$

So whether a is even or odd, $\delta((a^b)) \leq a+3b-5$. □

References

- [1] E. Arbarello, M. Cornalba, P. Griffiths, J. Harris: *Geometry of Algebraic Curves, vol. 1*. Springer-Verlag (1985).
- [2] A. Del Centina: *Weierstrass points and their impact in the study of algebraic curves: a historical account from the “Lückensatz” to the 1970s*. Ann. Univ. Ferrara 54, 37-59 (2008).
- [3] D. Edidin: *Brill-Noether theory in codimension-two*. J. Algebraic Geom. 2(1), 25-67 (1993).
- [4] D. Eisenbud, J. Harris: *Existence, decomposition, and limits of certain Weierstrass points*. Invent. math. 87, 495-515 (1987).
- [5] D. Eisenbud, J. Harris: *Irreducibility of some families of linear series with Brill-Noether number -1* . Annales scientifiques de l’É.N.S. 4^e serie 22:1, 33-53 (1989).
- [6] D. Eisenbud, J. Harris: *Limit linear series: basic theory*. Invent. math. 85, 337-371 (1986).
- [7] P. Griffiths, J. Harris: *On the variety of special linear systems on a general algebraic curve*. Duke math. journal 47:1, 233-272 (1980).
- [8] P. Griffiths, J. Harris: *Principles of algebraic geometry*. Wiley-Interscience (1978).
- [9] R. Hartshorne: *Algebraic Geometry*. Springer-Verlag (1977).
- [10] J. Komeda: *On primitive Schubert indices of genus g and weight $g-1$* . J. Math. Soc. Japan 43:3, 437-445 (1991).
- [11] J. Harris, I. Morrison: *Moduli of Curves*. Springer-Verlag 1991.
- [12] B. Osserman: *A limit linear series moduli scheme*. Ann. Inst. Fourier (Grenoble) 56:4, 1165-1205 (2006).
- [13] N. Pflueger: *Existence and deformations of non-primitive Weierstrass points*. In preparation.