Amherst College
Department of Mathematics and Statistics

MATH 105 FINAL EXAM FALL 2018

NAME: Solutions

Read This First!

- Keep cell phones off and out of sight.
- Do not talk during the exam.
- You are allowed one page of notes, front and back. No other books, notes, calculators, cell phones, communication devices of any sort, webpages, or other aids are permitted.
- Please read each question carefully. Show **ALL** work clearly in the space provided. There is an extra page at the back for additional scratchwork.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable.

Grading - For Instructor Use Only

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	21	15	9	8	9	12	12	12	15	12	125
Score:											

1. [21 points] Evaluate each limit.

(a) $\lim_{x\to 7} \dfrac{x^2-8x+7}{x^2-6x-7}$ $= \lim_{x\to 7} \dfrac{\cancel{(x-7)}(x-1)}{\cancel{(x-7)}(x+1)} = \dfrac{7-1}{7+1} = \dfrac{6}{8}$

$= \boxed{3/4}$

(b) $\lim_{x\to 1} \dfrac{\sqrt{4-3x}-1}{x^2-1}$ $= \lim_{x\to 1} \dfrac{(\sqrt{4-3x}-1)\cdot(\sqrt{4-3x}+1)}{(x+1)(x-1)\cdot(\sqrt{4-3x}+1)}$

$= \lim_{x\to 1} \dfrac{(4-3x)-1}{(x+1)(x-1)(\sqrt{4-3x}+1)} = \dfrac{-3\cancel{(x-1)}}{(x+1)\cancel{(x-1)}(\sqrt{4-3x}+1)}$

$= \dfrac{-3}{2\cdot(\sqrt{1}+1)} = \boxed{-3/4}$

(c) $\lim_{x\to 4^-} \dfrac{|x-4|}{x^2-3x-4}$ $|x-4| = -(x-4)$ for $x<4$, so:

$= \lim_{x\to 4^-} \dfrac{-\cancel{(x-4)}}{(x+1)\cancel{(x-4)}} = \dfrac{-1}{5}$

$= \boxed{-1/5}$

(continued on reverse)

(d) $\lim_{x\to-\infty} \frac{3-4x-4x^2}{9+2x^2}$

$$= \lim_{x\to-\infty} \frac{3-4x-4x^2}{9+2x^2} \cdot \frac{1/x^2}{1/x^2}$$

$$= \lim_{x\to-\infty} \frac{3/x^2 - 4/x - 4}{9/x^2+2} = \frac{3/\infty - 4/(-\infty) - 4}{9/\infty + 2}$$

$$= \frac{-4}{2} = \boxed{-2}$$

(e) $\lim_{x\to\infty} \frac{x^3-10x^2}{5x^2+7}$

$$= \lim_{x\to\infty} \frac{x^3-10x^2}{5x^2+7} \cdot \frac{1/x^2}{1/x^2} = \lim_{x\to\infty} \frac{x-10}{5+7/x^2}$$

$$= \frac{\infty - 10}{5+7/\infty} = \frac{\infty - 10}{5+0} = \boxed{+\infty}$$

(f) $\displaystyle\lim_{x\to 8} \frac{\frac{x}{x+4} - \frac{x-4}{x-2}}{x^2 - 10x + 16}$

$$= \lim_{x\to 8} \frac{\frac{x}{x+4}\cdot\frac{x-2}{x-2} - \frac{(x-4)}{x-2}\cdot\frac{x+4}{x+4}}{(x-2)(x-8)}$$

$$= \lim_{x\to 8} \frac{\frac{x^2-2x-x^2+16}{(x+4)(x-2)}}{(x-2)(x-8)}$$

$$= \lim_{x\to 8} \frac{-2\cancel{(x-8)}}{(x+4)(x-2)(x-2)\cancel{(x-8)}} = \frac{-2}{12\cdot 6\cdot 6}$$

$$= -\frac{1}{6\cdot 6\cdot 6} = \boxed{-\frac{1}{216}}$$

(g) $\displaystyle\lim_{x\to 2^+} \frac{x^2 - x - 6}{x^2 + x - 6}$

$$= \lim_{x\to 2^+} \frac{(x+2)(x-3)}{(x-2)(x+3)} = \frac{4\cdot(-1)}{0^+\cdot 5} = \frac{-4}{0^+} = \boxed{-\infty}$$

2. [15 points] Evaluate the **derivative** of each function. You do not need to simplify your answer.

(a) $f(x) = (3x-7)\left(x^{1/3} + \frac{1}{x^4}\right)$

$$f'(x) = 3\cdot\left(x^{1/3} + \frac{1}{x^4}\right) + (3x-7)\cdot\left(\frac{1}{3}x^{-2/3} - 4\cdot\frac{1}{x^5}\right)$$

(b) $g(x) = \left(2x^3 + 5x^4\right)^{1/3}$

$$g'(x) = \frac{1}{3}\cdot\left(2x^3 + 5x^4\right)^{-2/3}\cdot\left(6x^2 + 20x^3\right)$$

(c) $h(x) = (x^2+7)\sqrt{5x+3}$

$$h'(x) = 2x\sqrt{5x+3} + (x^2+7)\cdot\frac{1}{2\sqrt{5x+3}}\cdot 5$$

(continued on reverse)

(d) $f(x) = \dfrac{\sqrt{2x+3}}{x^2+1}$

$$f'(x) = \frac{\frac{1}{2\sqrt{2x+3}}2\cdot(x^2+1) - \sqrt{2x+3}\cdot 2x}{(x^2+1)^2}$$

(e) $g(x) = (2x-1)^3(5x+3)^5$

$$g'(x) = 3(2x-1)^2\cdot 2\cdot (5x+3)^5 + (2x-1)^3\cdot 5(5x+3)^4\cdot 5$$

3. [9 points] Let $f(x) = \dfrac{2x}{3x+1}$. Compute $f'(x)$ using the **limit definition** of the derivative. You may use the quotient rule to check your answer, but for full points all steps of the limit calculation must be shown.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{2(x+h)}{3(x+h)+1} - \frac{2x}{3x+1}}{h} = \lim_{h\to 0} \frac{\frac{2(x+h)(3x+1) - 2x[3(x+h)+1]}{[3(x+h)+1]\cdot(3x+1)}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{6x^2} + \cancel{2x} + \cancel{6xh} + 2h - \cancel{6x^2} - \cancel{6xh} - \cancel{2x}}{h(3(x+h)+1)(3x+1)}$$

$$= \lim_{h\to 0} \frac{2\cancel{h}}{\cancel{h}(3(x+h)+1)(3x+1)}$$

$$= \frac{2}{(3(x+0)+1)(3x+1)}$$

$$= \boxed{\frac{2}{(3x+1)^2}}$$

4. Consider the curve defined by the equation

$$y^2 = x^3 - x + 1.$$

(a) [4 points] Determine $\frac{dy}{dx}$ using **implicit differentiation.** Your answer will be in terms of both x and y.

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(x^3 - x + 1)$$

$$2y \cdot \frac{dy}{dx} = 3x^2 - 1$$

$$\boxed{\frac{dy}{dx} = \frac{3x^2-1}{2y}}$$

(b) [4 points] Find the equation of the tangent line at the point $(3, 5)$.

@$(3,5)$, $\quad \frac{dy}{dx} = \frac{3 \cdot 3^2 - 1}{2 \cdot 5} = \frac{26}{10} = 13/5$

So the tangent line is

$$y - 5 = \frac{13}{5}(x-3)$$

ie. $\quad y = \frac{13}{5}x - \frac{39}{5} + 5$

ie. $\quad \boxed{y = \frac{13}{5}x - \frac{14}{5}}$

5. [9 points] Find the absolute maximum and absolute minimum values of $f(x) = x^2(x-5)^3$ on the interval $[0, 6]$.

$$f'(x) = 2x(x-5)^3 + x^2 \cdot 3(x-5)^2$$
$$= x(x-5)^2 \cdot [2(x-5) + 3x]$$
$$= x(x-5)^2(5x-10)$$
$$= 5x(x-5)^2(x-2)$$

$\Rightarrow$ crit. numbers $0, 2, 5$.

Closed interval method:

$f(0) = 0^2 \cdot (-5)^3 = 0$

$f(2) = 2^2 \cdot (-3)^3 = 4 \cdot (-27) = -108$ ← min

$f(5) = 5^2 \cdot 0^3 = 0$

$f(6) = 6^2 \cdot 1^3 = 36$ ← max

$$\boxed{\begin{array}{l}\text{abs. max. is } 36 \text{ @ } x=6 \\ \text{abs. min. is } -108 \text{ @ } x=2.\end{array}}$$

6. [12 points] Consider the following function.

inc/dec/min 4
conc./infl. 4
asy. 4

$$f(x) = \frac{2-x}{x^2}$$

Sketch the graph $y = f(x)$. Clearly label the following features on your graph: asymptotes (horizontal or vertical), intervals where it is increasing/decreasing, intervals where it is concave up/down, local max(s) and min(s), and inflection point(s).

Division by 0 occurs @ $x=0$, & numerator is $2-0 \neq 0$ there

$\Rightarrow$ V.A. @ $x=0$.

$$\left.\begin{array}{l}\lim_{x\to\infty} \frac{2-x}{x^2} = \lim_{x\to\infty}\left(\frac{2}{x^2}-\frac{1}{x}\right) = \frac{2}{\infty}-\frac{1}{\infty} = 0 \\ \& \lim_{x\to-\infty} \frac{2-x}{x^2} = \frac{2}{(\infty)} - \frac{1}{-\infty} = 0\end{array}\right\} \text{so H.A. @ } y=0$$

$$f'(x) = \left(\frac{2}{x^2}-\frac{1}{x}\right)' = -\frac{4}{x^3} - (-1)\frac{1}{x^2} = -\frac{4}{x^3}+\frac{1}{x^2} = \frac{x-4}{x^3}$$

num. 0 @ $x=4$
denom 0 @ $x=0$

	$(-\infty,0)$	$(0,4)$	$(4,\infty)$
$x-4$	$-$	$-$	$+$
$\frac{1}{x^3}$	$-$	$+$	$+$
f'	$+$	$-$	$+$
f	↗	↘	↗

$\Rightarrow$
- inc on $(-\infty,0)$ & $(4,\infty)$
- dec. on $(0,4)$
- Local min @ $x=4$, $y=f(4)=-1/8$
- no extremum @ $x=0$ (it's a discontinuity).

$$f''(x) = \left(-\frac{4}{x^3}+\frac{1}{x^2}\right)' = \frac{12}{x^4} - \frac{2}{x^3} = \frac{12-2x}{x^4} = -2\cdot\frac{x-6}{x^4}$$

num. 0 @ $x=6$
denom 0 @ $x=0$

	$(-\infty,0)$	$(0,6)$	$(6,\infty)$
$x-6$	$-$	$-$	$+$
$-\frac{2}{x^4}$	$-$	$-$	$-$
f''	$+$	$+$	$-$
f	∪	∪	∩

$\Rightarrow$
- conc. up on $(-\infty,0)$ & $(0,6)$
- conc. down on $(6,\infty)$
- inflection pt. @ $x=6$, $y=f(6)=-\frac{4}{36}=-1/9$.

(sketch on next page)

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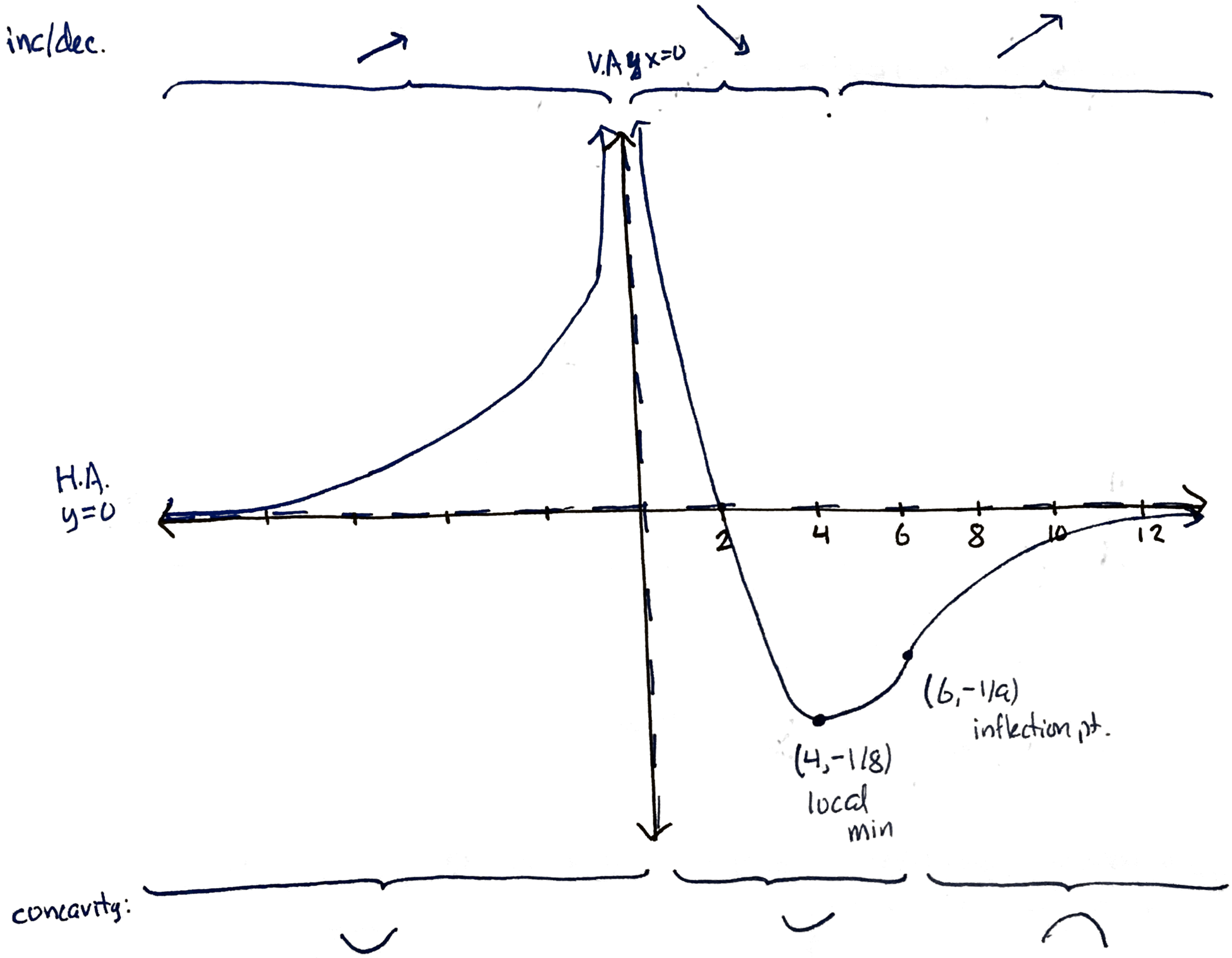
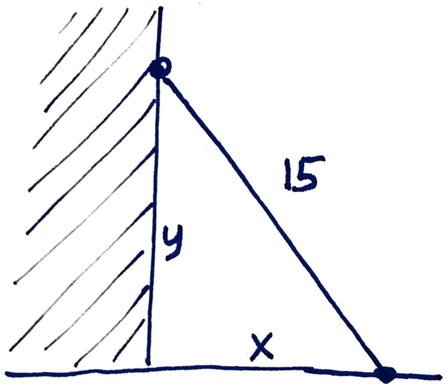
Sketch: (not exactly to scale)

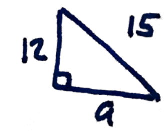


7. [12 points] A 15 foot ladder is leaning against a wall, and sliding down the side. At this moment, the bottom of the ladder is 9 feet away from the wall, and is sliding away at 2 feet per second. Determine how quickly the top of the ladder is sliding down the wall at this moment (in feet per second).



$$x^2+y^2=15^2$$

At key moment



$x=9$ $\quad y=\sqrt{15^2-9^2}=12$

$x'=2$ $\quad$ want: $y'=?$

$$\frac{d}{dt}(x^2+y^2)=\frac{d}{dt}(15^2)$$

$$\Rightarrow \quad 2x\cdot x'+2y\cdot y'=0$$

$\Rightarrow$ At key moment,

$$2\cdot 9\cdot 2+2\cdot 12\cdot y'=0$$

$$24y'=-36$$

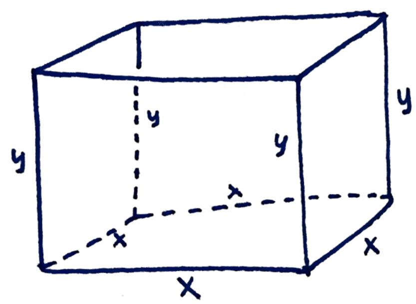
$$y'=-\frac{36}{24}=\underline{-3/2}$$

(neg. since decreasing as ladder falls).

$\boxed{\text{The top of the ladder is } \underline{\text{falling}} \text{ at } \underline{3/2} \text{ foot per second.}}$

8. [12 points] A small rectangular box with a square base **with no lid** is to be constructed out of 12 square inches of cardboard. Determine what dimensions the box should have in order for its volume to be as large as possible.

$\boxed{\text{Note: this is problem 3.7.15, which was on the homework (with smaller numbers to make the arithmetic easier).}}$



cardboard needed:

$$\underbrace{x^2}_{\text{base}} + \underbrace{4xy}_{\text{4 sides of area } xy} = 12 \quad \text{(constraint)}$$

we want to maximize volume:

$$V = x^2 y.$$

Solve for y using the constraint:

$$4xy = 12 - x^2$$
$$y = \frac{12-x^2}{4x}$$

So Volume, as a function of x, is $V(x) = x^{\cancel{2}}\left(\frac{12-x^2}{4\cancel{x}}\right) = \frac{1}{4}x(12-x^2)$

Feasible values: we need $x \geq 0$

& $y \geq 0 \Rightarrow \frac{12-x^2}{4x} \geq 0 \Rightarrow 12 \geq x^2 \Rightarrow x \leq \sqrt{12} = 2\sqrt{3}$

$\Rightarrow$ the interval is $[0, 2\sqrt{3}]$.

We can maximize w/ the closed interval method:

$$V(x) = \frac{1}{4}x(12-x^2)$$
$$V'(x) = \frac{1}{4}\cdot 1\cdot(12-x^2) + \frac{1}{4}x\cdot(-2x) = \frac{1}{4}(12 - x^2 - 2x^2)$$
$$= \frac{1}{4}(12-3x^2) = \frac{3}{4}(4-x^2)$$

$\Rightarrow$ crit. pts @ $x = \pm 2$; discard -2 since it's out of the interval.

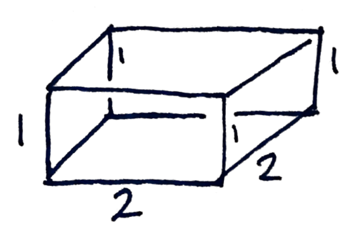
Check endpoints & $x=2$:

$V(0) = \frac{1}{4}\cdot 0\cdot 12 = 0$

$\underline{V(2)} = \frac{1}{4}\cdot 2\cdot 8 = 4 \leftarrow$ max

$V(2\sqrt{3}) = \frac{1}{4}\cdot 2\sqrt{3}\cdot 0 = 0$

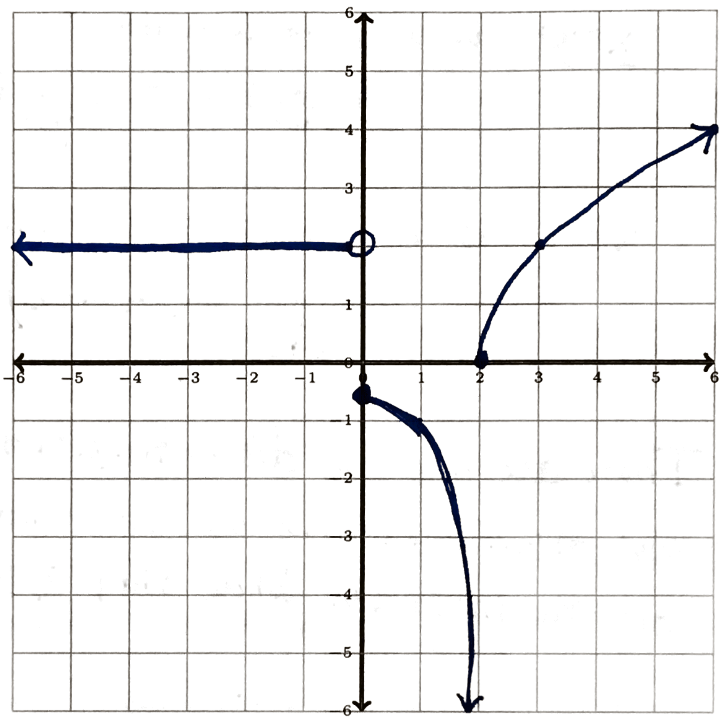
$\boxed{\text{So the max volume is 4, which occurs for } x=2,\ y = \frac{12-2^2}{4\cdot 2} = 1.}$



9. Consider the following piecewise function.

$$f(x) = \begin{cases} 2 & x < 0 \\ \frac{1}{x-2} & 0 \le x < 2 \\ 2\sqrt{x-2} & x \ge 2 \end{cases}$$

(a) [6 points] Sketch the graph $y = f(x)$ on the axes below (for this sketch, you don't need to take any derivatives or apply the techniques from Chapter 3; instead think about how each piece of the graph is obtained from a graph you already know about).



(b) [6 points] Evaluate each of the following quantities (no explanation or scratchwork is required).

- $\lim_{x \to 0^-} f(x) = 2$
- $\lim_{x \to 2^-} f(x) = -\infty$
- $\lim_{x \to 0^+} f(x) = -\frac{1}{2}$
- $\lim_{x \to 2^+} f(x) = 0$
- $f(0) = -\frac{1}{2}$
- $f(2) = 0$

(continued on reverse)

The definition of $f(x)$ is reproduced below for convenience.

$$f(x) = \begin{cases} 2 & x < 0 \\ \frac{1}{x-2} & 0 \leq x < 2 \\ 2\sqrt{x-2} & x \geq 2 \end{cases}$$

(c) [3 points] Determine all points where $f(x)$ is discontinuous.

$\underline{x=0}$ is a jump discontinuity

$\left(\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x) \right)$

$\underline{x=2}$ is an infinite discontinuity

$\left(\lim_{x \to 2^-} f(x) = -\infty \right)$

There are no other discontinuities; the three functions used are continuous on the intervals where they apply.

(continued on reverse)

10. [12 points] Consider the function

$$f(x) = \frac{x}{4+x^2}.$$

(a) Compute $f'(x)$, and simplify.

$$f'(x) = \frac{1\cdot(4+x^2) - x\cdot 2x}{(4+x^2)^2}$$

either of these is a fine way to leave the answer.

$$= \frac{4-x^2}{(4+x^2)^2}$$

$$= -\frac{(x+2)(x-2)}{(4+x^2)^2}$$

(b) Determine all critical numbers of $f(x)$.

denom. of $f'(x)$ is never 0 since $4+x^2 > 0$.

num. of $f'(x)$ is 0 @ $\boxed{x = \pm 2}$.

(continued on reverse)

(c) Determine the intervals on which $f(x)$ is increasing and decreasing.

$-2 \qquad 2$

$-\frac{1}{(4+x^2)^2}$	−	−	−
$x+2$	−	+	+
$x-2$	−	−	+
f'	−	+	−
f	↘	↗	↘

$\boxed{\text{dec. on } (-\infty,-2) \;\&\; (2,\infty) \quad \text{inc. on } (-2,2)}$

(d) Classify each critical point as a local minimum, a local maximum, or neither.

$x=-2$ is a local min. by 1st deriv. test (↘↗)

$x=+2$ is a local max by 1st deriv. test (↗↘).