Amherst College
Department of Mathematics and Statistics

MATH 105 TEST #3 (SLIGHTLY MODIFIED) FALL 2015

NAME: Solutions

Read This First!

- This is a closed-book examination. No books, notes, calculators, cell phones, communication devices of any sort, webpages, or other aids are permitted. Cell phones are to be out of sight.
- Please read each question carefully. Show $\boxed{\text{all}}$ work clearly in the space provided.
- If you need addition space to do a problem, please use the back of the $\boxed{\text{previous}}$ page.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable
- Answers must be clearly labeled.
- The exam consists of Questions 1–9, which total to 100 points.

Grading - For Instructor Use Only

Question:	1	2	3	4	5	6	7	Total
Points:	12	10	16	14	16	20	12	100
Score:								

1. [12 points] Let $f(x) = 3x^5 - 20x^3$.

   (a) Find the critical numbers of $f(x)$.

   $f'(x) = 15x^4 - 60x^2 = 15x^2(x^2-4) = 15x^2(x-2)(x+2)$

   $= 0$ when $x = 0$ or $x^2 = 4$, & never undefined

   ie crit. numbers are

   $\boxed{0, -2, \& +2.}$

   (b) Test whether they are local maximums, local minimums, or neither.

	-2		0		2	
$15x^2$	+		+	+		+
$x-2$	−		−	−		+
$x+2$	−		+	+		+
f'	+		−	−		+
f	↗		↘	↘		↗

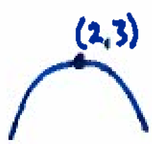
   by first deriv. test,

   There's a $\boxed{\text{local max at } x=-2 \;\&\; \text{local min at } x=+2.}$

   (no max/min at $x=0$)

2. [10 points] Consider a function $g(x)$ with the property that $g(2) = 3$, $g'(2) = 0$, and $g''(2)$ is some nonzero number. We are also given that $g(x)$ has a local maximum when $x = 2$. Determine whether $g''(2)$ positive is negative. Your solution should include an explanation (in words!) and a picture. Be sure to indicate how the words relate to the picture.

Because there's a local max. @ $x=2$, the second deriv. test says that $f''(2) < 0$ (since it's nonzero).



In other words, the graph must be concave down ~~X~~ rather than concave up at a local maximum.

3. [16 points] The radius and height of a cylinder are changing with respect to time. The radius is increasing at a rate of 2 cm/sec, while the height is decreasing at a rate of 1 cm/sec. How fast is the volume of the cylinder changing at the instant of time when the radius is 10 cm and the height is 5 cm? (You may assume that the volume of a cylinder of radius r and height h is $V = \pi r^2 h$.)

$V = \pi r^2 h$

$\Rightarrow V' = \pi \cdot 2r \cdot r' h + \pi r^2 \cdot h'$

<u>At key moment</u>:

$r' = 2$ (cm/sec)

$h' = -1$ (cm/sec)

$r = 10$ (cm)

$h = 5$ (cm)

substituting values at key moment:

$V' = \pi \cdot 2 \cdot 10 \cdot 2 \cdot 5 + \pi \cdot 10^2 \cdot (-1)$

$= 200\pi - 100\pi$

$= \boxed{100\pi}$ (cm/sec)

4. [14 points] Find the absolute maximum and minimum values of the function

$$f(x) = x(x-4)^3$$

on the interval $[0, 3]$.

$$f'(x) = 1\cdot(x-4)^3 + x\cdot 3(x-4)^2$$
$$= (x-4)^2\cdot[(x-4)+3x]$$
$$= (x-4)^2\cdot[4x-4]$$
$$= 4(x-4)^2(x-1)$$

This is never undefined.
It is 0 @ $x=1$ & $x=4$: only $x=1$ is in $[0,3]$.

Candidates: $x=0, x=1, x=3$.

$f(0) = 0\cdot(-4)^3 = 0$ ← max

$f(1) = 1\cdot(-3)^3 = -27$ ← min

$f(3) = 3\cdot(-1)^3 = -3$

max value is 0 @ $x=0$
min value is -27 @ $x=1$

5. [16 points] Find where $g(x) = \dfrac{x}{(x+3)^2}$ is increasing and decreasing.

$$g'(x) = \frac{1\cdot(x+3)^2 - x\cdot 2(x+3)}{[(x+3)^2]^2}$$

$$= \frac{(x+3)^{\cancel{2}} - 2x\cancel{(x+3)}}{(x+3)^{\cancel{4}3}}$$

$$= \frac{(x+3)-2x}{(x+3)^3} = \frac{3-x}{(x+3)^3}$$

sign could change at $x=3$ or $x=-3$ (num. or denom. 0)

	-3		3	
$3-x$	$+$	$+$	$-$
$\frac{1}{(x+3)^3}$	$-$	$+$	$+$
f'	$-$	$+$	$-$
f	↘	↗	↘

Decreasing on $(-\infty,-3)$ & $(3,\infty)$
Increasing on $(-3,3)$

6. [20 points] The function $f(x) = \dfrac{1+x^2}{x^2-4}$ has first and second derivatives given by:

$$f'(x) = \frac{-10x}{(x^2-4)^2}, \qquad f''(x) = \frac{10(3x^2+4)}{(x^2-4)^3}.$$

(a) Use this information to determine where $y = f(x)$ is increasing or decreasing, and find any local max(s) or local min(s).

Sign of f' could change at $x = \pm 2$ or 0 (num. or denom 0)

Note: $x = \pm 2$ are discontinuities of $f(x)$, so they cannot be local max(s) or mim.

	-2		0		2	
$-10x$	$+$		$+$	$-$		$-$
$\frac{1}{(x^2-4)^2}$	$+$		$+$	$+$		$+$
f	$\nearrow$		$\nearrow$	$\searrow$		$\searrow$

max

$\boxed{\text{increasing: } (-\infty,-2) \ \& \ (-2,0) \quad \text{decreasing: } (0,2) \ \& \ (2,\infty) \quad \text{Local max @ } x=0}$

(discontinuous @ $x=-2$)
(discontinuous @ $x=2$)

(b) Use this information to determine where $y = f(x)$ is concave up or concave down.

Sign of $f''(x)$ could change only at $x = \pm 2$ (where denom. is 0), since $3x^2+4 > 0$ for all x.

can factor:
$$\frac{10(3x^2+4)}{(x^2-4)^3} = \frac{10(3x^2+4)}{[(x+2)(x-2)]^3} = \frac{10(3x^2+4)}{(x+2)^3(x-2)^3}$$

		-2		2	
$10(3x^2+4)$	$+$		$+$		$+$
$\frac{1}{(x+2)^3}$	$-$		$+$		$+$
$\frac{1}{(x-2)^3}$	$-$		$-$		$+$
f''	$+$		$-$		$+$
f	$\cup$		$\cap$		$\cup$

$\boxed{\text{conc. up on } (-\infty,-2) \ \& \ (2,\infty) \quad \text{conc. down on } (-2,2)}$

(c) Compute $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. Use these values to identify any *horizontal asymptotes* of the graph $y = f(x)$.

$$\lim_{x\to\infty} \frac{1+x^2}{x^2-4}\cdot\frac{1/x^2}{1/x^2} = \lim_{x\to\infty}\frac{1/x^2+1}{1-4/x^2} = \frac{1/\infty^2+1}{1-4/\infty^2}$$

$$= \frac{0+1}{1-0} = \boxed{1}$$

$$\& \lim_{x\to-\infty}\frac{1+x^2}{x^2-4} = \lim_{x\to-\infty}\frac{1/x^2+1}{1-4/x^2}$$

$$= \frac{\frac{1}{(-\infty)^2}+1}{1-4/(-\infty)^2} = \frac{\frac{1}{\infty}+1}{1-4/\infty} = \frac{0+1}{1-0}$$

$$= \boxed{1}$$

$\boxed{\text{Horiz asymptote at } y=1}$

7. [12 points] The previous problem stated that $f(x) = \dfrac{1+x^2}{x^2-4}$ has derivatives

$$f'(x) = \frac{-10x}{(x^2-4)^2}, \qquad f''(x) = \frac{10(3x^2+4)}{(x^2-4)^3}.$$

Verify that these formulas are correct by computing $f'(x)$ and $f''(x)$ using the usual rules of differentiation.

$$f'(x) = \frac{2x(x^2-4) - (1+x^2)\cdot 2x}{(x^2-4)^2}$$

$$= \frac{\cancel{2x^3} - 8x - 2x - \cancel{2x^3}}{(x^2-4)^2}$$

$$= \frac{-10x}{(x^2-4)^2} \quad \checkmark$$

$$f''(x) = \frac{-10\cdot(x^2-4)^2 - (-10x)\cdot 2(x^2-4)2x}{[(x^2-4)^2]^2}$$

$$= \frac{\cancel{(x^2-4)}\cdot[-10(x^2-4) + 40x^2]}{(x^2-4)^{\cancel{4}3}}$$

$$= \frac{10\cdot[-x^2+4+4x^2]}{(x^2-4)^3}$$

$$= \frac{10\cdot(3x^2+4)}{(x^2-4)^3} \quad \checkmark$$