Derivatives

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1 Introduction

Last week, we discussed various notions of rates of change, including the so-called instantaneous rate of change, which is like an idealized average rate of change (averaged over a very small interval). Since then we've been discussing the idea of a *limit*, which takes some function and defined a sort of ideal value for a function. This lecture brings these two ideas together to define, formally, the instantaneous rate of change of a function. The resulting idea is called a *derivative* of the function. We define some notation for this idea and examine several basic examples.

The derivative is a fundamental idea in calculus. In my mind, it's most important role is to serve as helpful language: the word "derivative" has a precise meaning, which allow you to transfer intuition from one situation to another. This language allows physicist to speak very precisely about velocities and acceleration; indeed Newton defined the derivative with precisely this purpose in mind. The language also allows economists to speak precisely about growth (or decline) of economic indicators, and is used by biologists to speak precisely about growing and shrinking populations. The purpose today is to introduce this precise language and get a basic handle on what exactly it means.

2 The definition of the derivative

Given a function $f(x)$, and any value c, we define a number denoted $f'(c)$, which is called the *derivative of* f at c. The notation $f'(c)$ is often pronounced "f prime of c"; the word "prime" here has nothing to do with prime numbers; it just means "first" (for the first derivative). These definitions will range from informal to formal.

I'll illustrate each example using the function $f(x) = x^2$ at the value $c = 1$. The derivative is $f'(1) = 2$, which we'll see in various ways.

The first definition is perhaps the most physical interpretation: we image the function as telling us where an object is at some time, and ask about how quickly it is moving at the instant c.

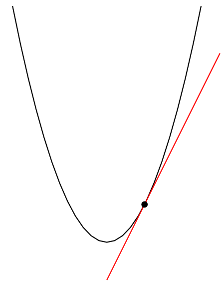
Definition 2.1. The derivative of f at c, $f'(c)$ is the the *instantaneous rate of change* of the function f around the value c.

For example, if $f(x) = x^2$, we know that $f(1) = 1$. We can compute that $f(1.01) = 1.0201, f(1.02 = 1.0404$, and $f(1.03) = 1.0609$. Rounding to the nearest hundredth, $f(1.01) \approx 1.02 f(1.02) \approx 1.04$, and $f(1.03) \approx 1.06$. So the rate that this function is increasing is roughly twice the rate that the input increases, and this principle only gets more accurate for smaller changes in input. So the instantaneous rate of change is 2.

We saw last week that the idea of an instantaneous rate of change is the same as a geometric notion, the slope of the tangent line.

Definition 2.2. The derivative $f'(c)$ is equal to the slope of the tangent line to the graph of $f(x)$ at $x = c$.

For example, if $f(x) = x^2$, we can draw the graph very carefully and then draw the line that perfectly kisses the graph at $(1, 1)$. This line turns out to have slope precisely 2.



Both of the definitions above can be thought of in terms of perfect limits of nearby imperfect objects.

Definition 2.3. The derivative $f'(c)$ is the limit as $x \to c$ of the slope of the secant line form $(c, f(c))$ to $(x, f(x))$.

For example, if $f(x)) = x^2$, then such a secant line passes through $(1,1)$ and (x, x^2), so its slope is $\frac{x^2-1}{x-1} = x+1$. The function $s(x) = x+1$ is continuous, so its limit as $x \to 1$ is $s(1) = 1+1 = 2$. So the "ideal" slope of a secant line is 2.

Finally, we come to the usual, mathematically formal definition of derivative.

Definition 2.4. The derivative $f'(c)$ is equal to the limit $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$.

This definition is the easiest to use algebraically, but I suggest that you always think about the other definitions as well when you are trying to reason conceptually.

3 Examples

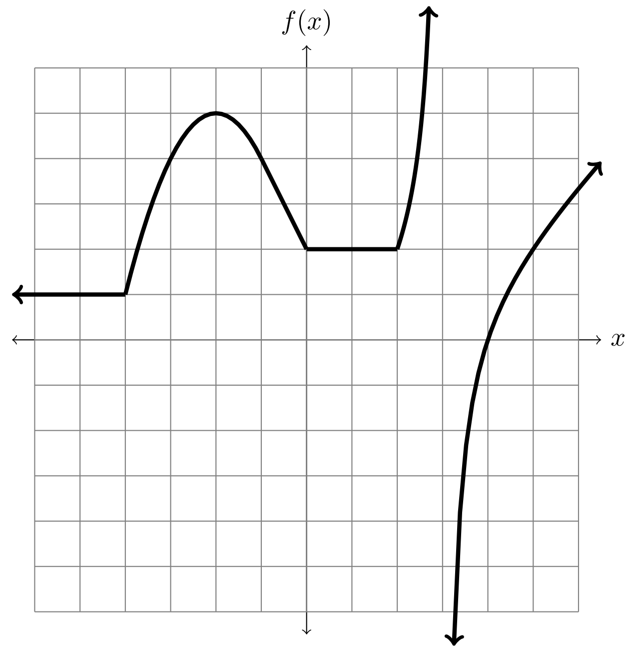
Example 3.1. Suppose that you are driving on a freeway. You pass mile marker 47 at 1:10pm, and set the cruise control of your car at this time. You reach mile marker 131 at 2:40pm. If $f(t)$ denotes the number of miles you have driven in your car exactly t hours after noon that day, what is $f'(2)$?

Solution. The value $f'(2)$ should be the instantaneous speed of the car at exactly 2:00pm. Since the cruise control is on, we can assume that the car travelled at a constant speed from 1:10pm to 2:40pm, a period of 90 minutes, or 1.5 hours. During that time, you traveled $131 - 47 = 84$ miles, therefore your average speed was $\frac{84}{1.5} = 63$ miles per hour. Therefore your instantaneous speed at 2:00pm should also be very close to 63 miles per hour. So $f'(2) = 63$.

Example 3.2. Recall that the area of a circle with radius r is πr^2. You can view this as a the value $A(r)$, where $A(x)$ is the function $A(x) = \pi x^2$. What is $A'(r)$? Interpret this using geometry.

Solution. We can use the formal definition of derivative. $A'(r) = \lim_{x\to r} \frac{\pi x^2 - \pi r^2}{x-r} = \lim_{x\to r} \frac{\pi(x+r)(x-r)}{x-r}$. Canceling on the top and bottom, this is $A'(r) = \lim_{x\to r} \pi(x+r) = 2\pi r$. You may recognize this as the formula for the circumference of a circle. The geometric interpretation of this fact is that as a circle's radius grows, the rate at which its area is growing is equal to the circumference. This makes sense visually: the circle is growing only at the boundary, whose size is measured by the circumference.

Example 3.3. Suppose that $f(x)$ is the function whose graph is sketched below. Assume that the grid lines are spaced 1 unit apart. Identify all values of c where $f'(c)$ is zero. For which values of c is $f'(c)$ positive? For which values is it negative? For which values of c (of those shown) does $f'(c)$ not exist? (Note that of course the picture isn't perfect; make your best guess as to what is going on, clearly stating your assumptions).



Solution. Interpret the derivative as the slope of a tangent line. Then $f'(c)$ should be 0 for values of c where the tangent line has slope 0, that is, lies horizontal. This certainly occurs where the graph itself is perfectly flat: all values $c < -4$ and also all values c in $(0, 2)$. In addition to these, the tangent line also lies flat at $(-2, 5)$ (you can see that the grid line itself is the tangent line at that point). I am assuming that the graph has a vertical asymptote at $x = 3$, and hence is not defined at that point.

The tangent line has positive slope where the graph is moving up and right: that is, for c in $(-4, -2)$, $(2, 3)$ and $c > 3$. So $f'(c) > 0$ for these values of c.

The tangent line has negative slope for c in $(-2, 0)$, where the graph is slanting downward to the right.

Finally, the derivative $f'(c)$ is note defined wherever the secant lines don't converge to a tangent line. This occurs for $c = -4$, $c = 0$, and $c = 2$; at these values the limit of secant lines from one side is perfectly horizontal, whereas the limit from the other side is not. So $f'(c)$ is not defined for these values of c.

Example 3.4. Let $f(x) = \sqrt{x} + 1$. What is $f'(c)$, where c is a constant?

Solution. Use the limit definition. $f'(c) = \lim_{x \to c} \frac{(\sqrt{x}+1) - (\sqrt{c}+1)}{x - c} = \lim_{x \to c} \frac{\sqrt{x} - \sqrt{c}}{x - c}$. There is an algebraic trick you can use to reformulate this expression, called *rationalizing the numerator.* The idea is to multiply by something on the top and the bottom that will remove the square roots from the top. What works in this case is to notice that $\frac{\sqrt{x}-\sqrt{c}}{x-c} = \frac{\sqrt{x}-\sqrt{c}}{x-c} \cdot \frac{\sqrt{x}+\sqrt{c}}{\sqrt{x}+\sqrt{c}} = \frac{x-c}{(x-c)(\sqrt{x}+\sqrt{c})} = \frac{1}{\sqrt{x}+\sqrt{c}}$. From this it follows that $f'(c) = \lim_{x \to c} \frac{1}{\sqrt{x} + \sqrt{c}} = \frac{1}{2\sqrt{c}}$.

Example 3.5. Let $f(x) = \frac{x+3}{x+4}$. What is $f'(0)$? What is $f'(c)$, in terms of the constant c? For which values c does this derivative exist?

Solution. The value $f'(0)$ is equal to the following limit. (Note: you do not need to show all of these steps in your work. I am more thorough here so that you can see all the moving parts).

$$
\begin{aligned}
f'(0) &= \lim_{x\to 0}(\text{Slope of secant line from } (0,f(0)) \text{ to } (x,f(x))) \\
&= \lim_{x\to 0}\frac{f(x)-f(0)}{x-0} \\
&= \lim_{x\to 0}\frac{\frac{x+3}{x+4}-\frac{3}{4}}{x} \\
&= \lim_{x\to 0}\frac{\frac{4(x+3)-3(x+4)}{4(x+4)}}{x} \\
&= \lim_{x\to 0}\frac{4(x+3)-3(x+4)}{4x(x+4)} \\
&= \lim_{x\to 0}\frac{x}{4x(x+4)} \\
&= \lim_{x\to 0}\frac{1}{4(x+4)} \\
&= \frac{1}{4(0+4)} \\
&= \frac{1}{16}
\end{aligned}
$$

The general case of $f'(c)$ for an arbitrary value c is no more difficult, but the notation is messier. It works as follows.

$$
\begin{aligned}
f'(c) &= \lim_{x\to c}\frac{f(x)-f(c)}{x-c} \\
&= \lim_{x\to c}\frac{\frac{x+3}{x+4}-\frac{c+3}{c+4}}{x-c} \\
&= \lim_{x\to c}\frac{(x+3)(c+4)-(x+4)(c+3)}{(x+4)(c+4)(x-c)} \\
&= \lim_{x\to c}\frac{xc+4x+3c+12-xc-3x-4c-12}{(x+4)(c+4)(x-c)} \\
&= \lim_{x\to c}\frac{x-c}{(x+4)(c+4)(x-c)} \\
&= \lim_{x\to c}\frac{1}{(x+4)(c+4)} \\
&= \frac{1}{(c+4)^2}
\end{aligned}
$$

The only way this limit could fail to exist is if $c=-4$, in which case we would be dividing by 0 at some of the steps. Of course, the function $f(x)$ is not defined at $x=-4$ anyway. So in other words **the derivative exists for all values of c except $c=-4$** – it exists wherever the function $f(x)$ is defined.

4 The equation of the tangent line

We've talked a great deal about the *slope* of the tangent line to the graph of a function, but we haven't really ever written down the equation of the line itself. I'll briefly describe how to do this.

Let's first do the example of $f(x)=x^2$, and find the tangent line at $x=c$. As we saw repeatedly in the first section, this tangent line has slope 2. That means that for any other point (x,y) on the tangent line,

$$\begin{aligned} \text{slope} &= \frac{\text{rise}}{\text{run}} \\ &= \frac{y-1}{x-1} \\ &= 2. \end{aligned}$$

This equation now gives the equation of the tangent line: $(y-1) = 2(x-1)$. If you put this in usual slope-intercept from, you get $y = 2x - 1$.

If you examine how we found the tangent line in the example above, you'll see that it generalizes perfectly. The equation of the tangent line is simply

$$(y - f(c)) = f'(c)(x - c)$$

Or if you prefer slope-intercept form, the equation of the tangent line is:

$$y = f'(c)x + (f(c) - cf'(c))$$

Example 4.1. Let $f(x) = x^2$.

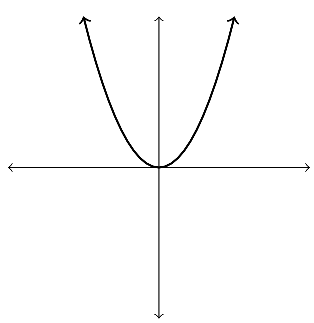
(a) Sketch the graph of $f(x)$

(b) Suppose that c and d are two constants. Find the *equation* of the secant line through $(a, f(a))$ and $(b, f(b))$.

(c) Find the equation of the tangent line through $(a, f(a))$.

Solution.

(a)



(b) The line through (a, a^2) and (b, b^2) as slope $\frac{b^2-a^2}{b-a}$. This simplifies to $\frac{(b+a)(b-a)}{(b-a)} = b+a = a+b$. Therefore as above, the equation of this line is $\frac{y-a^2}{x-a} = a+b$. Simplifying this equation gives $y-a^2 = (a+b)(x-a)$, or $y - a^2 = (a+b)x - (a+b)a$, or $\boxed{y = (a+b)x - ab}$.

(c) We can use the formula above. The derivative $f'(c)$ can be computed as a limit $\lim_{x\to c} \frac{x^2-c^2}{x-c} = \lim_{x\to c}(x+c) = 2c$. Using the formula right before this example, the equation of the tangent line is $y = 2cx + (c^2 - c \cdot 2c)$, or $\boxed{y = (2c)x - c^2}$.

Notice that the answer to part (c) is *exactly the same* as what we would get if we set both a and b to c in the answer to part (b). This makes sense: the tangent line is like a secant line, except that you imagine moving the two points together so that they really become the same point.

Example 4.2. Let $f(x) = \frac{1}{\sqrt{x}}$. What is the equation of the tangent line to the graph $y = f(x)$ at the point $(9, f(9))$?

Solution. The slope of a *secant* line from $(9, f(9))$ to some other point $(x, f(x))$ on the graph is

$$\frac{f(x) - f(9)}{x - 9} = \frac{1/\sqrt{x} - 1/3}{x - 9}.$$

To find the slope of the tangent line, we need to take the limit as x goes to 9. To do this requires a bit of algebra, including the trick of rationalizing the numerator.

$$\begin{aligned}
\lim_{x \to 9} \frac{1/\sqrt{x} - 1/3}{x - 9} &= \lim_{x \to 9} \frac{\frac{3}{3\sqrt{x}} - \frac{\sqrt{x}}{3\sqrt{x}}}{x - 9} \\
&= \lim_{x \to 9} \frac{3 - \sqrt{x}}{3\sqrt{x}(x - 9)} \\
&= \lim_{x \to 9} \frac{3 - \sqrt{x}}{3\sqrt{x}(x - 9)} \frac{3 + \sqrt{x}}{3 + \sqrt{x}} \\
&= \lim_{x \to 9} \frac{9 - x}{3\sqrt{x}(x - 9)(3 + \sqrt{x})} \\
&= \lim_{x \to 9} \frac{-1}{3\sqrt{x}(3 + \sqrt{x})} \\
&= \frac{-1}{3 \cdot 3 \cdot (3 + 3)} \\
&= -\frac{1}{54}.
\end{aligned}$$

Observe that this number is negative, which informs you that the tangent line is sloped downward. It is also fairly small. You should take a moment to think about why this is.

For the equation of the tangent line, we may write $y - f(9) = -\frac{1}{54}(x - 9)$, i.e. $y - \frac{1}{3} = -\frac{1}{54}x + \frac{1}{6}$. Moving the $\frac{1}{3}$ to the other side gives $y = -\frac{1}{54}x + \frac{1}{2}$.
