What f' says about f

25 September 2017

1 Introduction

Today we reverse some of the facts we saw last time, to infer information about a function f form its derivative function f'. Essentially, static information about $f'(x)$ becomes dynamic information about $f(x)$: when $f'(x)$ is positive, you know $f(x)$ is increasing; when $f'(x)$ is increasing, you know that $f(x)$ is curling upward; we will say that it is concave up. Finally, we'll discuss a couple things that it could mean about the graph of $f(x)$ if the derivative $f'(x)$ cannot be defined at a certain point.

2 Summary of key points

This points will all come up, and be shown in examples, in the proceeding discussion.

- The derivative $f'(x)$ determines $f(x)$ only up to vertical translation.

- A function f is increasing at $x \Leftrightarrow f'(x) > 0$.

- A function f is decreasing at $x \Leftrightarrow f'(x) < 0$.

- A function f is called "concave up" at $x \Leftrightarrow f''(x) > 0$.

- A function f is called "concave down" at $x \Leftrightarrow f''(x) < 0$.

- If a function is differentiable, then it is continuous. In other words, if it is discontinuous, then it is non-differentiable.

- A function can be continuous at x without being differentiable at x.

3 Sketching f from f'

To illustrate how knowledge about $f'(x)$ informs us about $f(x)$, consider the following example. Suppose that we know the graph of the derivative $f'(x)$ of some function looks as follows.

$y = f'(x)$
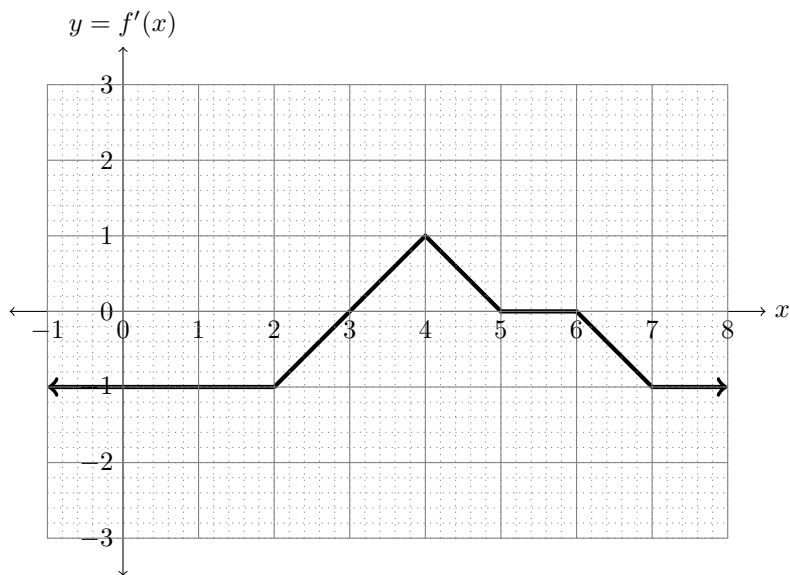
Let's attempt to sketch what the graph $f(x)$ could look like. I will draw it, in stages, as a red curve on the same plot as the black curve.
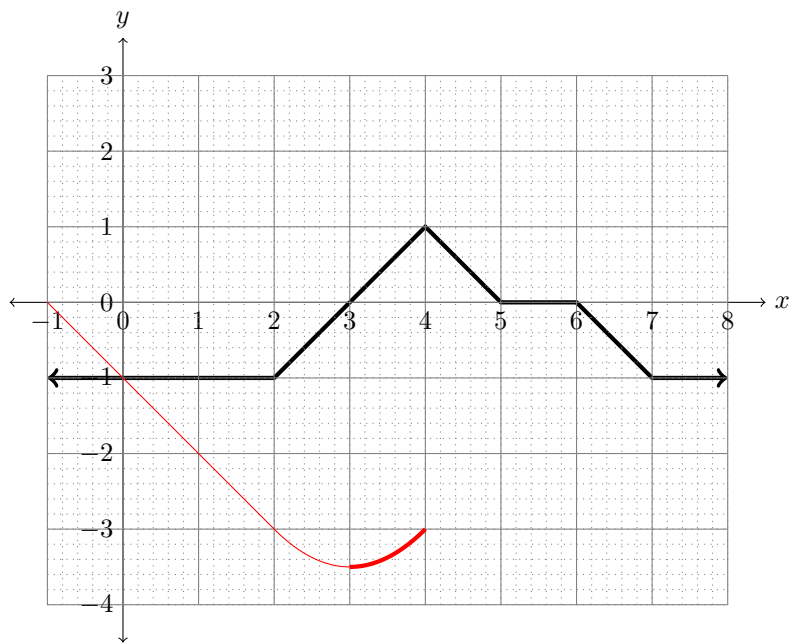
First, notice that $f'(x) = -1$ for all x from -1 to 2. So on these values, $f(x)$ will simply be a straight line of slope -1.
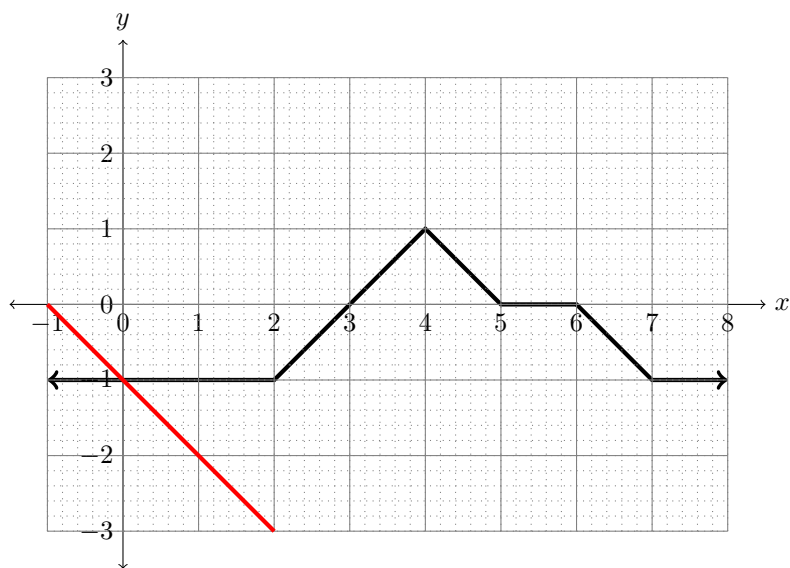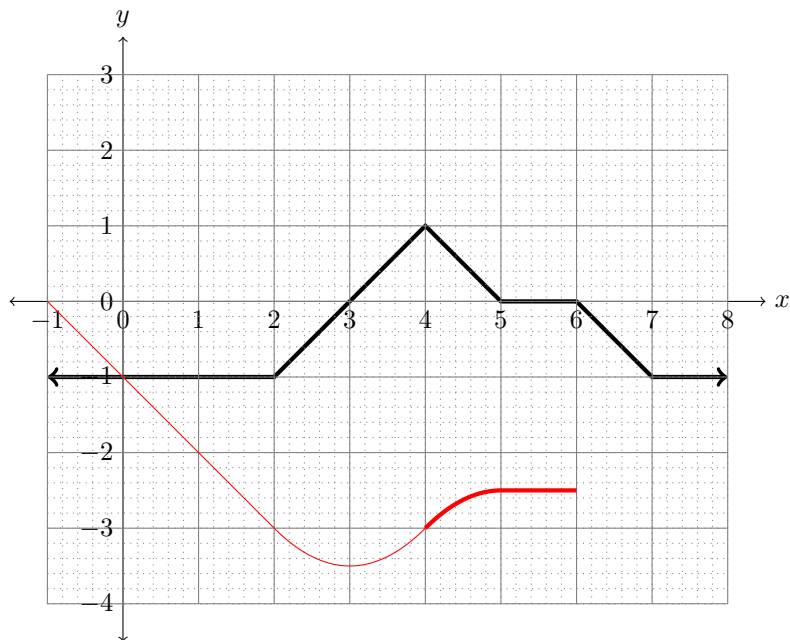

Now, we know that by the time $x = 3$, the tangent line to $y = f(x)$ must have slope $f'(3) = 0$. From $x = 2$ to $x = 3$ the slope of the tangent line will increase towards 0.
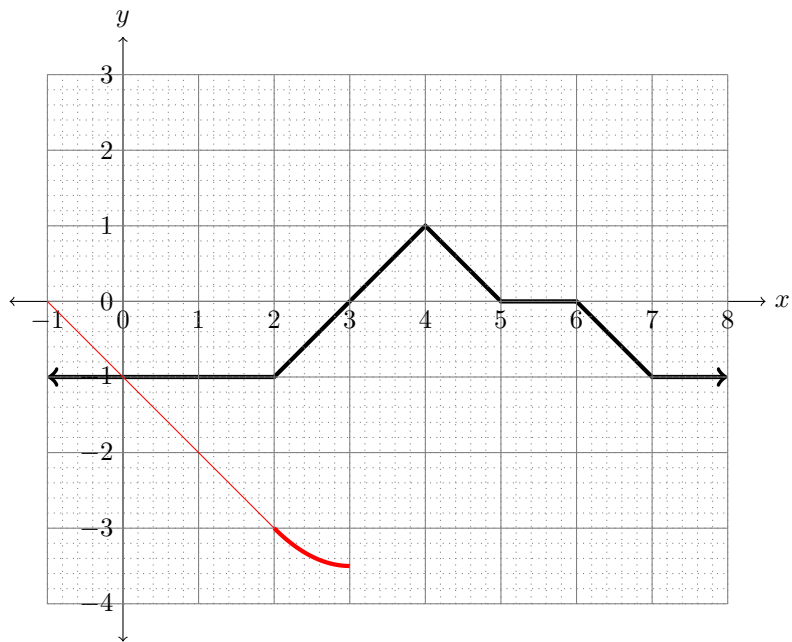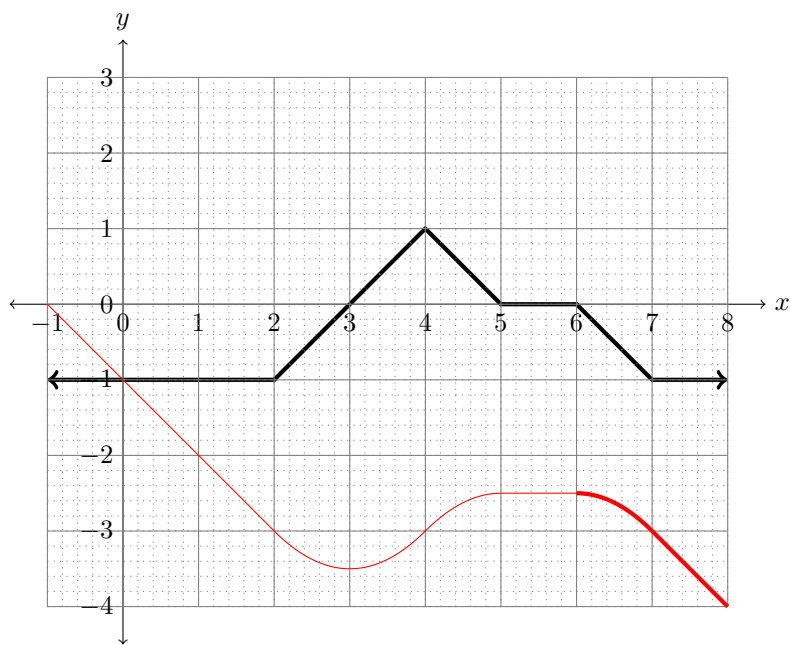
After that, the slope will continue increasing until it gets to 1 at $x = 4$.

Then the curve will turn again, back towards being flat (horizontal tangent line). From $x = 5$ to $x = 6$, $f'(x) = 0$, so in fact the graph will be flat in this interval.

Then the graph will slant back down, eventually to slope -1 at $x = 7$, and then it will proceed as a line of slope -1, the way it began.

But here's a question: how did I know to start drawing the graph of $f(x)$ at the origin $(0, 0)$? The answer is I didn't. Each of the red curves below could equally well be the graph of $f(x)$, for example.
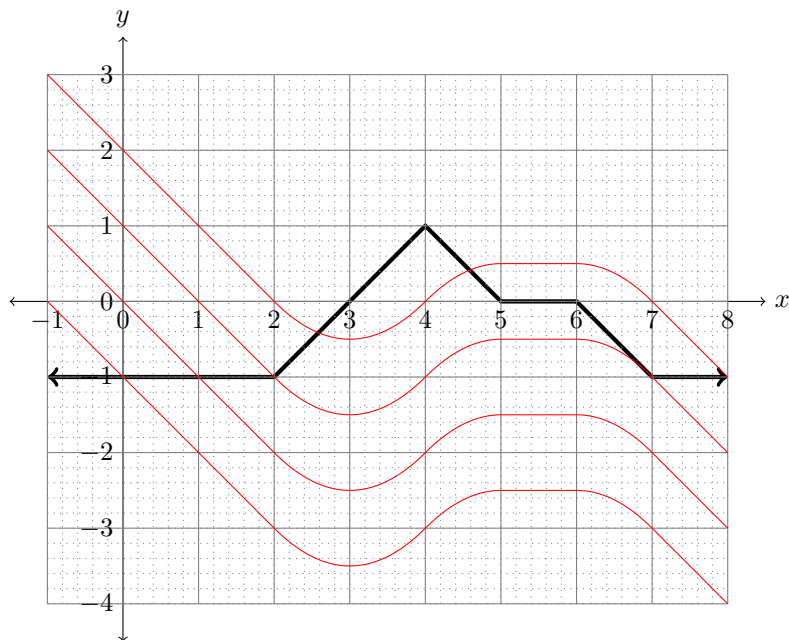
The point is that **the derivative cannot tell two graphs apart if one is just a vertical translation of the other**.

The other important thing to observe in this discussion is that the places where f is increasing wherever $f'(x) > 0$, and decreasing wherever $f(x) < -0$. The graph appears to lie flat where $f'(x) = 0$ on the nose. These principles are true in general.

4 Concavity

Another thing to observe in the example above is that when the graph of $f'(x)$ is increasing, the graph of $f(x)$ appears to "curl upwards." Now, "$f'(x)$ is increasing" just means the same thing as $f''(x) > 0$. At a point where $f''(x) > 0$, we say that the graph of $f(x)$ is *concave up.* Similarly, at points where $f''(x) < 0$, we say that $f(x)$ is *concave down.* For example, here is the function from the first section, with its first and second derivative shown. Concave up parts of $f(x)$ are shown in red, concave down parts are shown in blue. The other parts, where the graph of $f(x)$ is linear and therefore not concave in either direction, are drawn in green.
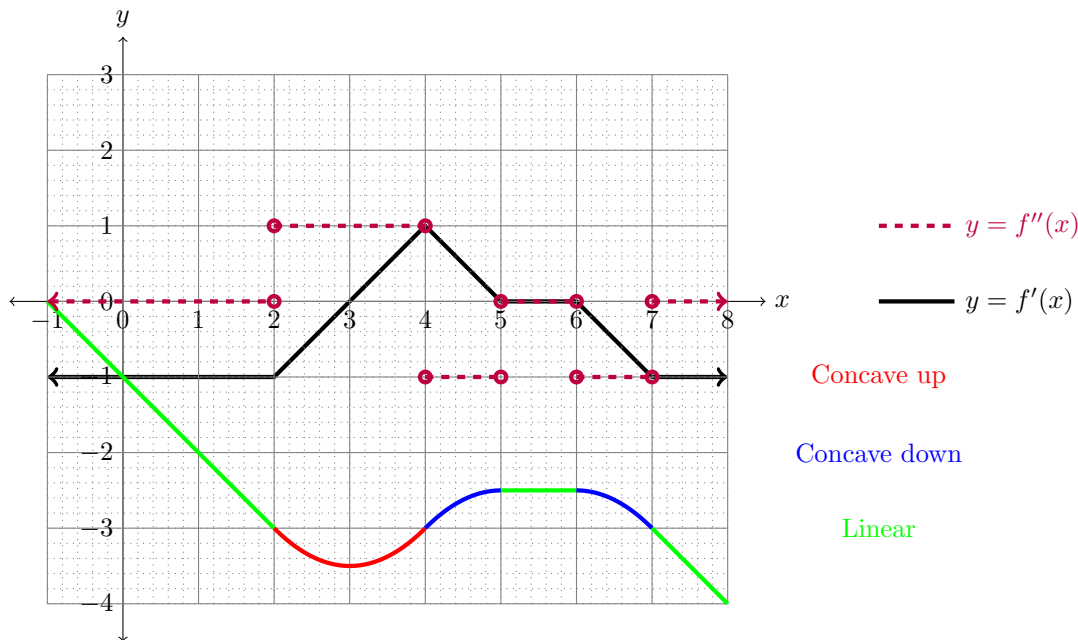
The concavity of a function has a number of qualitative interpretations. I'll mention a couple of them.

Example 4.1. **Diminishing returns.** In economics, one often talks about diminishing returns. This means, roughly, the more money you put into something, the less good you get from it per dollar. A classic example is: the more money you earn at your job, the less happy each additional dollar will make you. This means that your "utility curve" looks something like this.
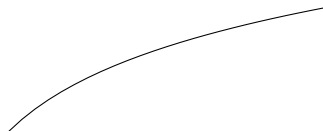

In other words: something has diminishing returns if the returns are **concave down** as a function of cost.

Example 4.2. **Population growth.** One of the main reasons that people worry about overpopulation is the following assumption: that that more people there are on Earth, the faster the population will grow (since there are more people to have children). In particular, people describe population as a J-shaped curve. A mathematically precise way to state with worry is: population is **concave up** as a function of time. (It is worth noting that this assumption is not always valid; there may be other forces that cause people to have fewer children).

Example 4.3. **Acceleration.** If $f(t)$ is position (e.g. distance from a given point on a road) as a function of time, then the graph of f is concave up if the vehicle is accelerating, while it is concave down if the vehicle is decelerating.

5   Examples

Example 5.1. Suppose that a function $f(x)$ has the following graph (from Monday's notes).

$y = f(x)$
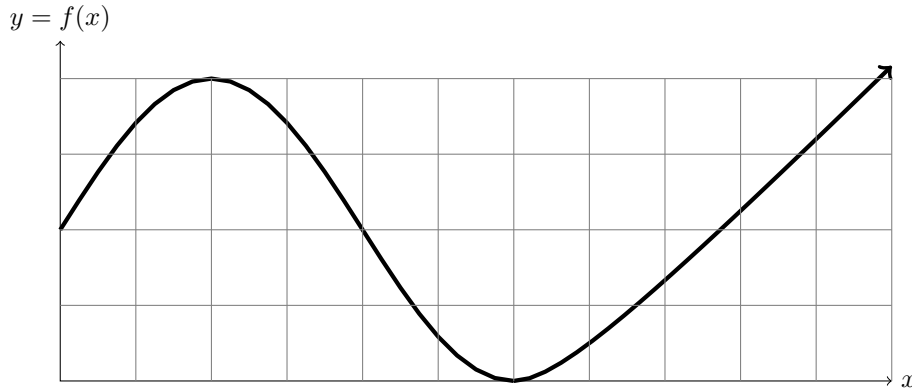
For which values of x is $f'(x)$ positive? For which values of x is $f''(x)$ negative?

Solution. The derivative $f'(x)$ is positive where $f(x)$ is increasing: we see that this is on $(0, 2)$ and $(6, \infty)$. The second derivative $f''(x)$ is positive where the graph is concave up. It is hard to pick out exactly where this is, but in fact it is on $(0, 4)$.

6 Non-differentiable points

The derivative $f'(c)$ at a particular value $x = c$ is defined by a limit. Like any other limit, this could fail to exist. When it does, the function is called non-differentiable. There are many reasons that a function could have a non-differentiable point. I'll just mention a couple.

First, **if the function isn't continuous at x, then it isn't differentiable at x.**

Example 6.1. Consider the function $f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$. It's graph looks like this:
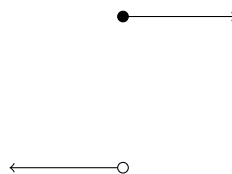

If you try to evaluate the derivative $f'(0)$, you will need to evaluate $\lim_{h \to 0} \dfrac{f(h) - 1}{h}$. In fact, this limit does not exist. Indeed, if the limit from the right does exist: it is $\lim_{h \to 0^+} \dfrac{1 - 1}{h} = \lim_{h \to 0} \dfrac{0}{h} = 0$. But the limit form the left does not exist: it is $\lim_{h \to 0^-} \dfrac{-1 - 1}{h} = \lim_{h \to 0^-} \dfrac{-2}{h}$, which does not converge, due to a vertical asymptote. Since one of the one-sided limits doesn't exist, the two-sided limit certainly doesn't exist.

A second reason that a point might not be differentiable is if the slopes of the secant lines approach different limits form the left and the right. We'll call such a situation a **corner** (I've heard other people call it a "cusp," although I tend not to use that term since it means something somewhat different in my field of research).

Example 6.2. Let $f(x) = |x|$ (the absolute value of x). Then $f'(0) = \lim_{h \to 0} \dfrac{|x|}{x}$. Now notice that $\dfrac{|h|}{h} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$. So in particular, $\lim_{h \to 0^-} \dfrac{|h|}{h} = -1$ but $\lim_{h \to 0^+} \dfrac{|h|}{h} = 1$. The two one-sided limits disagree, so the two-sided limit does not exist. In other words, $f'(0)$ does not exist. Visually, this is because there

are two "limits of secant lines" (shown with dotted lines below), one from each side; there is not one single tangent line.
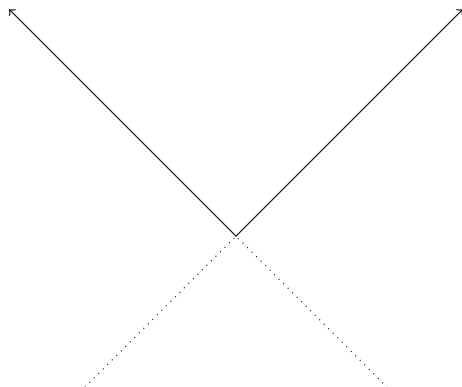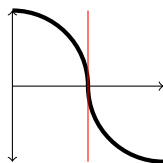
A third reason that a point might not be differentiable is if there is a **vertical tangent line** at that point (the slope of a vertical line is not well-defined[1]).

Example 6.3. Let $f(x) = \begin{cases} \sqrt{1 - x^2} & \text{if } 0 \leq x \leq 1 \\ -\sqrt{1 - (x - 2)^2} & \text{if } 1 < x \leq 2 \end{cases}$, defined for x in $[0, 2]$. The graph of this function is shown below.
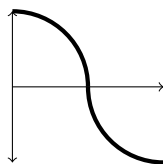
Then the tangent line to this graph at $x = 1$ is the vertical line $x = 1$.

This vertical line does not have a well-defined slope, so the function $f(x)$ is not differentiable at $x = 0$.

If you compute the slopes of secant lines algebraically and attempt to take the limit to the tangent line, you will see that the slope of the line blows up to negative ∞.

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[1]Really, it should just be defined to be ∞, but there are good reasons to think of this as a different sort of situation.