Products and quotients

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1 Introduction

This lecture catalogs two more rules that can be used to efficiently differentiate a larger family of functions (without appeal to the limit definition). We will discuss how to differentiate a product or quotient of any two functions (given that we can differentiate the two original functions).

2 The product rule

The product rule has a fairly simple form.

$$\boxed{\frac{d}{dx}f(x)g(x) = \frac{df}{dx}g(x) + f(x)\frac{dg}{dx}}$$

There are many other ways to write it. Another favorite is this rather concise alternative.

$$\boxed{(fg)' = f'g + fg'}$$

The way I usually remember the product rule is with the following explanation: the change in $f(x)g(x)$ is the sum of the change due to $f(x)$ and the change due to $g(x)$. If it were just $f(x)$ changing, then the derivative would just be $f'(x)g(x)$ ($g(x)$ would behave like a constant), while if it were just $g(x)$ changing it would be the reverse. The total change of $f(x)g(x)$ simply has both these contributions.

A nice illustration of the product rule (and a good way to confirm that it makes sense) is to see what is says about the function x^2. This is just the same thing as $x \cdot x$, so the product rule easily computes that $\frac{d}{dx}x^2 = \frac{dx}{dx}x + x\frac{dx}{dx} = x + x = 2x$. To my mind, this is the actually the best way to understand why the derivative of x^2 is $2x$ (rather than, say, x, or $3x$).

Here are a couple more examples.

Example 2.1. Consider $f(x) = (x^7 + 1)(x^9 + 1)$. There are two ways to differentiate this function; both are easy in different ways (and it is often smart to try to do both methods, as a way of checking your work). One method is the product rule: $f'(x) = (7x^6)(x^9+1)+(x^7+1)(9x^8) = 7x^{15}+7x^6+9x^{15}+9x^8 = 16x^{15}+9x^8+7x^6$. Another is to begin by expanding the function out to $f(x) = x^{16} + x^9 + x^7 + 1$, and differentiate this with the power rule to obtain the same answer.

Example 2.2. Consider $f(x) = (x + 1)(\sqrt{x} + 1)$. The derivative of this function can be computed with the product rule as follows.

$$
\begin{aligned}
\frac{d}{dx}\left[(x+1)(\sqrt{x}+1)\right] &= \left[\frac{d}{dx}(x+1)\right]\cdot(\sqrt{x}+1)+(x+1)\cdot\left[\frac{d}{dx}(\sqrt{x}+1)\right] \\
&= 1\cdot(\sqrt{x}+1)+(x+1)\cdot\frac{1}{2\sqrt{x}} \\
&= \sqrt{x}+1+\frac{1}{2}\sqrt{x}+\frac{1}{2\sqrt{x}} \\
&= \frac{3}{2}\sqrt{x}+1+\frac{1}{2\sqrt{x}}
\end{aligned}
$$

Observe that in this case, there is also an alternative way to compute the same derivative: we could expand out the product $(x+1)(\sqrt{x}+1)$ as $x\sqrt{x}+x+\sqrt{x}+1$, and differentiate the four terms of this sum individually.

3 The quotient rule

The quotient rule has a somewhat more nauseating form than the product rule. The usual expression is the following (both of these are equivalent, just using different notation for the derivative).

$$\boxed{\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{df}{dx}g(x)-f(x)\frac{dg}{dx}}{g(x)^2}}$$

$$\boxed{\left(\frac{f}{g}\right)' = \frac{f'g-fg'}{g^2}}$$

,

3.1 Alternative forms and derivations

Personally, I find it much easier, instead of trying to remember these sorts of ugly formulas, to just remember one special case, which is easy to remember since it closely resembles $\frac{d}{dx}\frac{1}{x} = -\frac{1}{x^2}$.

$$\left(\frac{1}{g}\right)' = -\frac{g'}{g^2}$$

From here, the quotient rule follows just by applying the product rule.

$$\left(f\cdot\frac{1}{g}\right)' = f'\cdot\frac{1}{g} - f\cdot\frac{g'}{g^2}$$

In fact, the equation above is the form in which I usually prefer to apply the quotient rule myself.

There is also a fairly quick derivation of the quotient rule from the product rule, as follows.

$$
\begin{aligned}
f &= g \cdot \frac{f}{g} \\
\Rightarrow f' &= \left(g \cdot \frac{f}{g} \right)' \\
&= g' \frac{f}{g} + g \left(\frac{f}{g} \right)' \\
\Rightarrow f' - g' \frac{f}{g} &= g \left(\frac{f}{g} \right)'
\end{aligned}
$$

Dividing both sides of this equation by g (and doing some mild simplifying) now gives the quotient rule.

3.2 Examples

One of the main uses of the quotient rule is to differentiate rational functions. For example:

$$
\begin{aligned}
\frac{d}{dx} \left(\frac{1}{x+1} \right) &= \frac{(\frac{d}{dx} 1)(x+1) - 1 \frac{d}{dx}(x+1)}{(x+1)^2} \\
&= -\frac{1}{(x+1)^2} \\
\frac{d}{dx} \left(\frac{2x+3}{x^2+1} \right) &= \frac{\left[\frac{d}{dx}(2x+3) \right] (x^2+1) - (2x+3) \frac{d}{dx}(x^2+1)}{(x^2+1)^2} \\
&= \frac{2(x^2+1) - (2x+3)(2x)}{(x^2+1)^2} \\
&= \frac{-2x^2 - 6x + 2}{(x^2+1)^2}
\end{aligned}
$$

Appendix: Deriving the product and quotient rules

In case you are interested, I have written below derivations of the product and quotient rules from the limit definition of the derivative.

The product rule can be derived by a slightly clever manipulation of the difference quotient used to define the derivative. The underlying idea is: we want to somehow create a factor of $f(x+h) - f(x)$ in the numerator, which we can factor out to realize $f'(x)$ in the limit. The argument is as follows.

$$
\begin{aligned}
(f \cdot g)'(x) &= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} g(x+h) + f(x) \frac{g(x+h) - g(x)}{h} \right] \\
&= \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \right] \lim_{h \to 0} [g(x+h)] + \lim_{h \to 0} [f(x)] \lim_{h \to 0} \left[\frac{g(x+h) - g(x)}{h} \right] \\
&= f'(x)g(x) + f(x)g'(x).
\end{aligned}
$$

As for the quotient rule, we have observed earlier in this lecture that it is enough to derive the formula $\frac{d}{dx}\frac{1}{g(x)} = \frac{g'(x)}{g(x)^2}$, since the general formula then follows by applying the product rule to $f(x) \cdot \frac{1}{g(x)}$. This special case can be derived as follows.

$$
\begin{aligned}
\frac{d}{dx}\frac{1}{g(x)} &= \lim_{h\to 0} \frac{\frac{1}{g(x+h)} - \frac{1}{g(x)}}{h} \\
&= \lim_{h\to 0} \frac{\frac{g(x)}{g(x)g(x+h)} - \frac{g(x+h)}{g(x)g(x+h)}}{h} \\
&= \lim_{h\to 0} \frac{g(x) - g(x+h)}{h \cdot g(x)g(x+h)} \\
&= \lim_{h\to 0} \frac{g(x) - g(x+h)}{h} \cdot \lim_{h\to 0} \frac{1}{g(x)g(x+h)} \\
&= -g'(x) \cdot \frac{1}{g(x)^2}.
\end{aligned}
$$

It is worth staring at this derivation carefully and taking note of where exactly the various aspects of the anatomy of the final formula arise. The minus sign comes because, when subtracting one reciprocal from another, the denominators are subtracted *in the reverse order* in the resulting numerator. The factor of $g(x)^2$ in the eventual denominator is the manifestation of the common denominator $g(x)g(x+h)$ that we needed to form in the first step.