Derivatives of trigonometric functions

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1 Introduction

Today we will then discuss the derivates of the six standard trigonometric functions. Of these, the most important are sine and cosine; the derivatives of all the other standard trigonometric follow readily from these.

2 Sine and cosine

We've seen how to differentiate polynomials and exponential functions. The other main missing piece in the catalog of elementary functions that can be easily differentiated are the trigonometric functions. The two basic facts are the following.

$$\begin{aligned} \frac{d}{dx}\sin x &= \cos x \\ \frac{d}{dx}\cos x &= -\sin x \end{aligned}$$

So these two basic functions are closely linked to each other; the main confusing thing is to remember which one obtains a negative sign when it is differentiated. The easiest way to get straight on this is just to think about where the functions are increasing and decreasing. The graph of sine is initially increasing, so its derivative at 0 had better by positive; thus is must be $\cos x$ and not $-\cos x$. Similarly, the graph of $\cos x$ starts at a local maximum, so its derivative but change from positive to negative around $x = 0$; this is the opposite of what $\sin x$ does, so the derivative of $\cos x$ must be $-\sin x$ and not $\sin x$.

Observation. Both sine and cosine have the very special property that they are the *negative* of their second derivative, i.e. $(\sin x)'' = -\sin x$ and $(\cos x)'' = -\cos x$. In physical terms, each curve is always accelerating back towards the x axis at a rate given by its distance from the x axis. This is the reason that these functions arise so much in physical problems: any system with "feedback" that pulls it back towards equilibrium (e.g. a weight on a spring, or a swaying bridge) is governed by some equations that ultimately give rise to functions that are built up from sine and cosine.

Note. The fact that the derivatives of sine and cosine have such a nice form in terms of each other is the principle reason why radians, rather than degrees, are always used when doing trigonometry (at least when any techniques from calculus are begin used). It is analogous to using the metric system in chemistry: just like the metric system makes unit conversions less error-prone, using radians makes taking derivatives less error-prone. This is exactly analogous to using e^x rather than any other exponential function[1].

[1] If you happen to have seen the formula, in terms of complex numbers, $e^{ix} = \cos x + i \sin x$, you will realize that choosing e and choosing radians are really the exact same choice, if you take a slightly broader point of view.

2.1 The derivation

The usual derivation of the derivatives of sine and cosine uses the following standard trigonometric identities (the word "identity" means a formula which hold for all values of the input). I've put everything involving h in blue to make it stand out.

$$\begin{aligned}\sin(x+h) &= \sin x \cos h + \cos x \sin h \\ \cos(x+h) &= \cos x \cos h - \sin x \sin h\end{aligned}$$

The basic facts that allow us to compute the derivatives of sine and cosine are the following *linear approximations* to $\sin x$ and $\cos x$: for x for *close to* 0,

$$\begin{aligned}\sin x &\approx x \\ \cos x &\approx 1\end{aligned}$$

The first of these was an identity we discussed in the lecture on linear approximation. The second follows because $\cos x$ has a local maximum at $x = 0$, hence a horizontal tangent line.

Applying these linear approximations and the identities above, we obtain the following fact: if h is a very small number (very close to 0), then:

$$\begin{aligned}\sin(x+h) &\approx \sin x \cdot 1 + \cos x \cdot h \\ \cos(x+h) &\approx \cos x \cdot 1 - \sin x \cdot h\end{aligned}$$

From these approximations, we see that $\sin x$ increases at a rate of $\cos x$ (as h increases), while $\cos x$ increases at a rate of $-\sin x$.

The more formal version of what I have just said is to first invoke the following two limits (which are both computed by some analysis using the squeeze theorem, which we will not describe in detail). Both are visually plausible if you draw the graphs of sine and cosine; they say that the derivative of sine at 0 is 1, and that the derivative of cosine at 0 is 0.

$$\begin{aligned}\lim_{h\to 0} \frac{\sin h}{h} &= 1 \\ \lim_{h\to 0} \frac{\cos h - 1}{h} &= 0\end{aligned}$$

Then the derivative of $\sin x$ can be computed as follows.

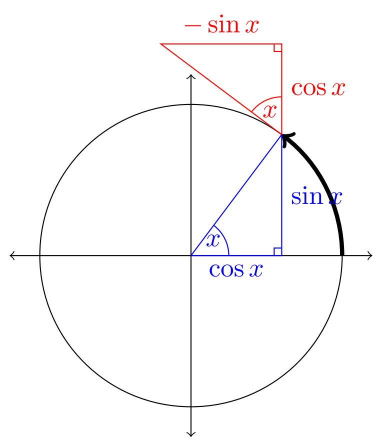
$$
\begin{aligned}
(\sin x)' &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}\\
&= \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h}\\
&= \lim_{h\to 0}\left(\sin x\frac{\cos h-1}{h}+\cos x\frac{\sin h}{h}\right)\\
&= \sin x\lim_{h\to 0}\frac{\cos h-1}{h}+\cos x\lim_{h\to 0}\frac{\sin h}{h}\\
&= \sin x\cdot 0+\cos x\cdot 1\\
&= \cos x
\end{aligned}
$$

The derivative of $\cos x$ can be formally commuted in a totally analogous way.

$$
\begin{aligned}
(\cos x)' &= \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}\\
&= \lim_{h\to 0}\frac{\cos x\cos h-\sin x\sin h-\cos x}{h}\\
&= \lim_{h\to 0}\left(\cos x\frac{\cos h-1}{h}-\sin x\frac{\sin h}{h}\right)\\
&= \cos x\lim_{h\to 0}\frac{\cos h-1}{h}-\sin x\lim_{h\to 0}\frac{\sin h}{h}\\
&= \cos x\cdot 0-\sin x\cdot 1\\
&= -\sin x
\end{aligned}
$$

2.2 A physical interpretation

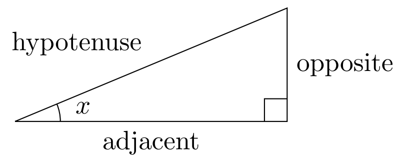
Skip this subsection if you don't particularly like physics. But I find the following picture to be the clearest explanation for why the derivatives of sine and cosine are what they are.



In this picture, you should imagine the curved arc as a planet orbiting the origin (1 unit away) in a perfect circle, traveling at speed exactly 1. Then the velocity of this orbiting planet will point in a direction tangent to the circle, and will have magnitude 1. Then you can determine the x and y coordinates of velocity by drawing the red triangle shown. It is congruent to the blue triangle, but rotated 90 degrees. Then the derivatives of sine and cosine can be seen by immediate visual inspection in this picture.

3 The six standard trigonometric functions

As you may have seen in your precalculus, there are six functions that usually make up the "standard trigonometric functions." As follows.



Function name	Notation	Definition
sine	$\sin x$	opposite / hypotenuse
tangent	$\tan x$	opposite / adjacent
secant	$\sec x$	hypotenuse / adjacent

Function name	Notation	Definition
cosine	$\cos x$	adjacent / hypotenuse
cotangent	$\cot x$	adjacent / opposite
cosecant	$\csc x$	hypotenuse / opposite

The nomenclature here is a bit of a nightmare, I'm afraid. All these terminology is couple hundred years old, and it one of those vestigial organs that we cannot seem to excise from common usage. However, this is the predominant nomenclature for these functions so it is worth reviewing them.

Here's one feature that makes all this slightly easier: these six functions are arranged into four "co-function" pairs: sine and cosine; tangent and cotangent; secant and cosecant. Each function is related to its "co-function" in a simple way: just swap "adjacent" and "opposite" wherever you see them[2].

All six can be expressed in terms of sine and cosine, as follows.

$$\begin{aligned}
\tan x &= \frac{\sin x}{\cos x} \\
\cot x &= \frac{\cos x}{\sin x} \\
\sec x &= \frac{1}{\cos x} \\
\csc x &= \frac{1}{\sin x}
\end{aligned}$$

As a result, it is straightforward to differentiate all of these functions by use of the quotient rule and the derivatives of sine and cosine. It is worth working these computations yourself, as an exercise in the quotient rule. The computations are shown below.

I suggest that you practice deriving these formulas, even if you ultimately memorize them. This is good practice with the quotient rule (which will pay dividends in more complex computations), and also will help you avoid overburdening your brain with arbitrary formulas.

$$\begin{aligned}
\frac{d}{dx}\tan x &= \frac{(\sin x)'\cos x - \sin x(\cos x)'}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} \\
&= \sec^2 x
\end{aligned}$$

[2] A more mathematical way to say this is: replace x with $\pi/2 - x$ (so $\cos(x) = \sin(\frac{\pi}{2} - x)$, $\cot x = \tan(\frac{\pi}{2} - x)$, and $\csc x = \sec(\frac{\pi}{2} - x)$.

$$
\begin{aligned}
\frac{d}{dx}\cot x &= \frac{(\cos x)' \sin x - \cos x (\sin x)'}{\sin^2 x} \\
&= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} \\
&= \frac{-1}{\sin^2 x} \\
&= -\csc^2 x
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dx}\sec x &= \frac{0 - 1 \cdot (\cos x)'}{\cos^2 x} \\
&= \frac{\sin x}{\cos^2 x} \\
&= \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} \\
&= \sec x \tan x
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dx}\csc x &= \frac{0 - 1 \cdot (\sin x)'}{\sin^2 x} \\
&= -\frac{\cos x}{\sin^2 x} \\
&= -\frac{1}{\sin x}\frac{\cos x}{\sin x} \\
&= -\csc x \cot x
\end{aligned}
$$

Note that there are generally many ways to write these derivatives (for example, the derivative of $\tan x$ could be written either $\frac{1}{\cos^2 x}$ or $\sec^2 x$). The convention is generally to write functions without denominators if possible; this entails replacing $\frac{1}{\cos x}$ with $\sec x$ where possible, for example.

By the way, here's a helpful mnemonic for remembering some of these derivatives, which I called informally in class the "co-function rule[3]." If you can remember the derivative of one function (e.g. if you remember that $\frac{d}{dx}\tan x = \sec^2 x$), then you can obtain the derivative of it's "co-function" as follows: take the derivative of the original function, replace all functions that appear by their co-functions, and then multiply by -1. For example, if you know that $\frac{d}{dx}\tan x = \sec^2 x$, then the "cofunction rule" tells you that $\frac{d}{dx}\cot x = -\csc^2 x$. If you look over the six derivative we have computed above, you will see that they all obey this rule.

[3]In fact, this rule can be deduced from the chain rule, I will leave you to think about why.