Implicit differentiation

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1 Introduction

Today we discuss a technique called implicit differentiation. Implicit differentiation is not a new differentiation rule; instead, it is a technique that can be applied with the rules we've already learned. The idea is this: instead of trying to tackle the desired function explicitly, instead just find a simple equation that the function satisfies (called an *implicit equation*. If you differentiate both sides of this equation, then you can usually recover the derivative of the function you actually cared about with just a little algebra.

2 Implicit and explicit functions

As a first step, I want to elaborate on what it means to write an implicit equation. An implicit equation is just an equation in two variables x and y that is not necessarily of the form $y = f(x)$. Here are some examples.

Explicit equations:

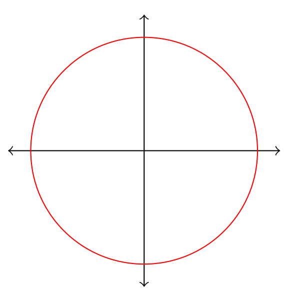
- $y = \sqrt{x}$
- $y = \sqrt{1 - x^2}$
- $y = \frac{x}{x-1}$

Implicit equations:

- $y^2 = x$
- $x^2 + y^2 = 1$
- $xy = x + y$

One major distinction between explicit and implicit equations is that implicit equations *don't necessarily describe graphs of functions.* Instead, they describe graphs of curves. Perhaps the simplest example is this implicit equation, which defines a circle.

$x^2 + y^2 = 1$ describes this curve:



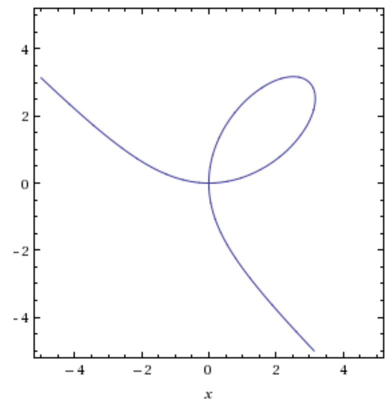
Now there are two graphs that lie on the circle above: $y = \sqrt{1-x^2}$ and $y = -\sqrt{1-x^2}$ (one is the upper semicircle, and one is the lower semicircle). So the *implicit* equation $x^2 + y^2 = 1$ describes two different *explicit* equations. This is often the case with implicit equations.

Here are two more examples of implicit equations, which we'll revisit in the last section.

Example 2.1. (Descartes' leaf) Consider the following equation.

$$x^3 + y^3 = 6xy$$

This is an implicit equation. It describes the curve shown below (made with Wolfram alpha).



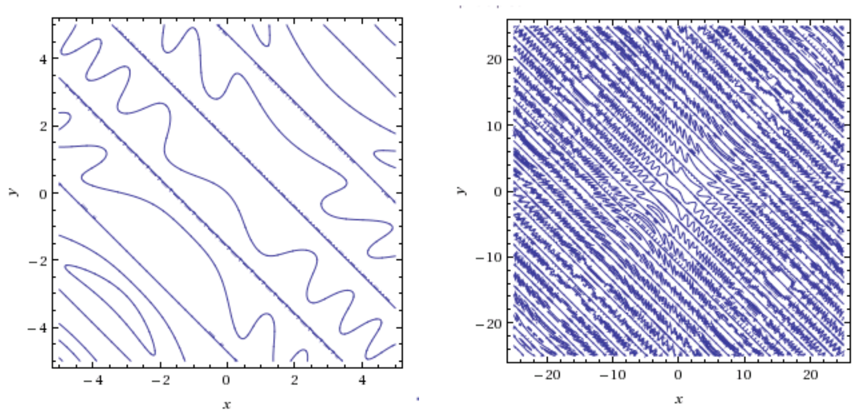
This curve is traditionally called the "folium of Descartes" ("folium" is Latin for "leaf"). This equation first occurred in a letter from Descartes to Fermat (Fermat was a lawyer by profession, but did mathematics as a hobby). Fermat claimed to have a method to find tangent lines to any curve, and Descartes invented this curve as a challenge to Fermat. Fermat was successful in finding tangent lines to the curve. Today, however, the problem of finding tangent lines is very easy, due to the invention of calculus some time later. This event was notable enough in the history of calculus that it is memorialized on the following Albanian postage stamp[1].

Example 2.2. Consider the following implicit equation.

[1] I am not aware of any relation between Descartes and Albania, but I only spent a couple minutes googling for it.

$$\tan(x+y) = \sin(xy)$$

If you graph its solution curve, it is the following very complex picture (shown at two different levels of zoom).



As you can see, there are *many* different functions that obey this implicit equation, because there are many different values of y for a given value of x.

3 Implicit differentiation

Now we'll look at how you can use an implicit equation to find derivatives. The basic technique will always be as follows.

- Differentiate both sides of the implicit equation. Remember that y **is a function of x**.
- Use algebra to solve for $\frac{dy}{dx}$. The result will be an expression in x and y.

Let's begin with a simple example.

Example 3.1. Consider the implicit equation of a circle.

$$x^2 + y^2 = 1$$

Now imagine that y is some function of x that obeys this implicit equation. We can differentiate both sides of the equation with respect to x.

$$\begin{aligned}
\frac{d}{dx}(x^2+y^2) &= \frac{d}{dx}1 \\
\frac{d}{dx}x^2 + \frac{d}{dx}y^2 &= 0 \\
2x + 2y \cdot \frac{dy}{dx} &= 0 \quad \text{(chain rule)} \\
2y\frac{dy}{dx} &= -2x \\
\frac{dy}{dx} &= -\frac{2y}{2x} \\
\frac{dy}{dx} &= -\frac{y}{x}
\end{aligned}$$

This shows that the slope of the tangent line to the circle described by $x^2 + y^2 = 1$ is always given by $-y/x$ at a point (x, y).

Note. When you solve for $\frac{dy}{dx}$, it will almost always be an expression in terms of both x and y. In some cases, you can re-express it as something purely in terms of x, but not always. The next example shows one case where you can re-express it.

Example 3.2. Suppose that $y = \sqrt[3]{\cos x + 7}$. Find $\frac{dy}{dx}$ using implicit differentiation.
Note. You can differentiate this using the chain rule as well. Of course you will get the same answer as we get below, but you may find one technique or the other easier.
Solution. Cube both sides to obtain $y^3 = \cos x + 7$. Differentiate both sides:

$$\begin{aligned}
\frac{d}{dx}y^3 &= \frac{d}{dx}(\cos x + 7) \quad \text{(differentiate both sides)} \\
3y^2\frac{dy}{dx} &= -\sin x \quad \text{(chain rule on the left, known derivatives on the right)} \\
\frac{dy}{dx} &= -\frac{\sin x}{3y^2} \quad \text{(divide both sides by } 3y^2\text{)}
\end{aligned}$$

In this case, we have an explicitly equation for y, namely $y = \sqrt[3]{\cos x + 7}$, so we can substitute that back into the answer here to get the derivative purely in terms of x.

$$\begin{aligned}
\frac{dy}{dx} &= -\frac{\sin x}{3\left(\sqrt[3]{\cos x + 7}\right)^2} \\
&= -\frac{\sin x}{3\left(\cos x + 7\right)^{3/2}}
\end{aligned}$$

To summarize, there are two main reasons to differentiate implicit equations.

1. Because you have no explicit equation for y in terms of x (you can still obtain $\frac{dy}{dx}$ in terms of x and y).
2. Because you have an explicit equation, but an implicit equation is much simpler (in this case, you can substitute your explicit equation back at the end to get $\frac{dy}{dx}$ purely in terms of x).

4 Examples

As examples of the technique of implicit differentiation, we will find some tangent lines to the two implicit curves given in section 2.

Example 4.1. Consider the equation for Descartes' leaf.

$$x^3 + y^3 = 6xy$$

1. Find the tangent line to this curve at the point $(3, 3)$.
2. Find the tangent line to this curve at the point $(\frac{4}{3}, \frac{8}{3})$.

Solution. Begin by differentiating both sides of the equation, as functions of x.

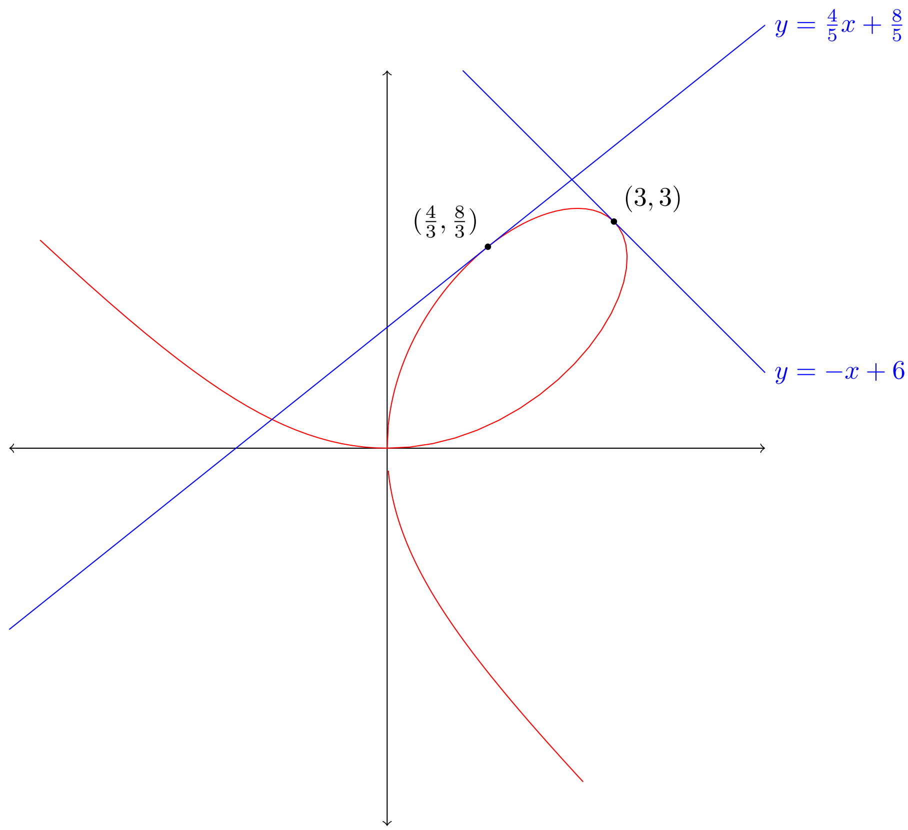
$$\begin{aligned}
\frac{d}{dx}\left(x^3+y^3\right) &= \frac{d}{dx}(6xy) \\
3x^2+3y^2\frac{dy}{dx} &= 6\frac{dx}{dx}y+6x\frac{dy}{dx} \quad \text{(product rule used on the right side)} \\
3x^2+3y^2\frac{dy}{dx} &= 6y+6x\frac{dy}{dx} \\
3y^2\frac{dy}{dx}-6x\frac{dy}{dx} &= 6y-3x^2 \quad \text{(move all the } \tfrac{dy}{dx} \text{ to one side)} \\
(3y^2-6x)\frac{dy}{dx} &= 6y-3x^2 \quad \text{(group like terms)} \\
\frac{dy}{dx} &= \frac{6y-3x^2}{3y^2-6x} \quad \text{(divide on both sides)} \\
\frac{dy}{dx} &= \frac{2y-x^2}{y^2-2x} \quad \text{(cancel the factor of 3 on top and bottom)}
\end{aligned}$$

Now, we can use this expression to compute the two desired tangent lines.

1. At the point $(3,3)$, the slope of the tangent line is $\frac{dy}{dx} = \frac{2\cdot 3-3^2}{3^2-2\cdot 3} = \frac{-3}{3} = -1$. So the tangent line is the line with slope -1 through the point $(3,3)$. Therefore this line is given by the equation $\boxed{(y-3)=(-1)(x-3)}$, or alternatively $\boxed{y=-x+6.}$

2. At the point $(\frac{4}{3},\frac{8}{3})$, the slope of the tangent line is given by $\frac{2\cdot\frac{8}{3}-(\frac{4}{3})^2}{(\frac{8}{3})^2-2\cdot\frac{4}{3}}$. This simplifies to $\frac{16/3-16/9}{64/9-8/3} = \frac{48/9-16/9}{64/9-24/9} = \frac{32/9}{50/9} = \frac{32}{40} = \frac{4}{5}$. So the equation of the tangent line is $\boxed{(y-\frac{8}{3})=\frac{4}{5}(x-\frac{4}{3})}$, or if you prefer, $\boxed{y=\frac{4}{5}x+\frac{8}{5}.}$

The curve, with these two tangent lines, is shown below.



Example 4.2. Consider the curve given by the following implicit equation.

$$\tan(x + y) = \sin(xy)$$

Find the tangent line to this curve at the point $(\sqrt{\pi}, -\sqrt{\pi})$.

Solution. Begin by differentiating both sides of the equation. As usual, regard y as a function of x.

$$\begin{aligned}
\tan(x+y) &= \sin(xy) \\
\frac{d}{dx}\tan(x+y) &= \frac{d}{dx}\sin(xy) \\
\sec^2(x+y)\frac{d}{dx}(x+y) &= \cos(xy)\frac{d}{dx}(xy) \quad \text{(chain rule on both sides)} \\
\sec^2(x+y)\left(\frac{dx}{dx} + \frac{dy}{dx}\right) &= \cos(xy)\left(\frac{dx}{dx}y + x\frac{dy}{dx}\right) \quad \text{(product rule on the right)} \\
\sec^2(x+y)\left(1 + \frac{dy}{dx}\right) &= \cos(xy)\left(y + x\frac{dy}{dx}\right) \quad \left(\text{because } \frac{dx}{dx} = 1\right)
\end{aligned}$$

So far so good. Now solve this whole monster for $\frac{dy}{dx}$.

$$\begin{aligned}
\sec^2(x+y) + \sec^2(x+y)\frac{dy}{dx} &= y\cos(xy) + x\cos(xy)\frac{dy}{dx} \quad \text{(distributing terms)} \\
\left(\sec^2(x+y) - x\cos(xy)\right)\frac{dy}{dx} &= y\cos(xy) - \sec^2(x+y) \quad \text{(moving all the } \frac{dy}{dx} \text{ to one side)} \\
\frac{dy}{dx} &= \frac{y\cos(xy) - \sec^2(x+y)}{\sec^2(x+y) - x\cos(xy)} \quad \text{(divide on both sides)}
\end{aligned}$$

To find the slope at the specific point $(\sqrt{\pi}, -\sqrt{\pi})$, just substitute the values of x and y into this expression. Notice first of all that $xy = -\pi$ and $x + y = 0$; these occur in multiple places in the expression.

$$\begin{aligned}
\text{slope} &= \frac{-\sqrt{\pi}\cos(-\pi) - \sec^2(0)}{\sec^2(0) - \sqrt{\pi}\cos(-\pi)} \\
&= \frac{-\sqrt{\pi}\cdot(-1) - 1}{1 - \sqrt{\pi}\cdot(-1)} \\
&= \frac{\sqrt{\pi} - 1}{\sqrt{\pi} + 1} \\
&\approx 0.28
\end{aligned}$$

Therefore the equation of the tangent line is $\boxed{(y + \sqrt{\pi}) = \frac{\sqrt{\pi}-1}{\sqrt{\pi}+1}\cdot(x - \sqrt{\pi})}$. In slope-intercept form, this is $\boxed{y = \frac{\sqrt{\pi}-1}{\sqrt{\pi}+1}x - \frac{2\pi}{\sqrt{\pi}+1}}$. If you use a calculator to approximate all these numbers, you can also write this equation as $\boxed{(y + 1.77) = 0.28\cdot(x - 1.77)}$ or $\boxed{y = 0.28x - 2.27}$ (any of these four boxed answers would be acceptable on homework). This line is shown on the plot below (Wolfram alpha has marked all the points of intersection; the point $(\sqrt{\pi}, -\sqrt{\pi})$ is the second from the right, where the line is tangent to the curve).