Solutions to Practice Test B for Midterm Exam 2

1. **(25 points)** Compute the following derivatives by any legal method. Simplify when asked to do so.

(a). Compute $f'(x)$, where $f(x) = x^2 \sec(3x-5)$. Simplify your answer.

(b). Compute $g''(x)$ (the *second* derivative), where $g(x) = \dfrac{x^2-4x+3}{\sqrt{x}}$. Simplify your answer.

(c). Compute $h(x) = \sqrt{x} + \dfrac{1}{\sqrt{x}} + \dfrac{1}{1+\sqrt{x}} + \dfrac{1}{\sqrt{1+x}}$. (Do not simplify your answer.)

(d). Compute $g'(x)$, where $g(x) = \left(\dfrac{1}{x^3}+\pi\right)^4 \cdot \left(x^4 - \dfrac{1}{x^7}\right)^{-5}$ (Do not simplify your answer.)

Solutions. (a) $f'(x) = 2x\sec(3x-5) + x^2\sec(3x-5)\tan(3x-5)\cdot 3 = \boxed{x\sec(3x-5)\big[2+3x\tan(3x-5)\big]}$

(b) $g(x) = x^{3/2} - 4x^{1/2} + 3x^{-1/2}$, so $g'(x) = \frac{3}{2}x^{1/2} - 2x^{-1/2} - \frac{3}{2}x^{-3/2}$,

and hence $\boxed{g''(x) = \frac{3}{4}x^{-1/2} + x^{-3/2} + \frac{9}{4}x^{-5/2}}$

(c) $h(x) = x^{1/2} + x^{-1/2} + (1+x^{1/2})^{-1} + (1+x)^{-1/2}$, so
$h'(x) = \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-3/2} - (1+x^{1/2})^{-2}\cdot\frac{1}{2}x^{-1/2} - \frac{1}{2}(1+x)^{-3/2}$

(d) We have $g(x) = \left(x^{-3}+\pi\right)^4\left(x^4-x^{-7}\right)^{-5}$, so

$$\boxed{g'(x) = 4\left(x^{-3}+\pi\right)^3\cdot\left(-3x^{-4}\right)\cdot\left(x^4-x^{-7}\right)^{-5} + \left(x^{-3}+\pi\right)^4\cdot(-5)\left(x^4-x^{-7}\right)^{-6}\cdot\left(4x^3+7x^{-8}\right)}$$

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2. **(15 points)** Find the **locations** of the **absolute maximum** and **absolute minimum value(s)** of the function

$$F(x) = (x-1)^2(2x-10)^2 \qquad \text{on the interval} \qquad [0,4].$$

Solutions. (By Closed Interval Method):
$F'(x) = 2(x-1)\cdot 1(2x-10)^2 + (x-1)^2\cdot 2(2x-10)\cdot 2 = 2(x-1)(2x-10)\big[(2x-10)+2(x-1)\big]$
$= 2(x-1)(2x-10)[4x-12] = 16(x-1)(x-5)(x-3)$,
which is **always defined**.

Solving $F' = 0$ gives $x = 1,3,5$, but 5 is not in the interval.

So the only critical numbers are $x = 1,3$. Testing them and endpoints:

$$F(0) = (-1)^2(-10)^2 = 100, \quad F(1) = 0, \quad F(3) = 2^2(-4)^2 = 64, \quad F(4) = 3^2(-2)^2 = 36.$$

The maximum value is 100, and the minimum value is 0.
So the $\boxed{\text{absolute max occurs at } x=0}$ and the $\boxed{\text{absolute min occurs at } x=1}$

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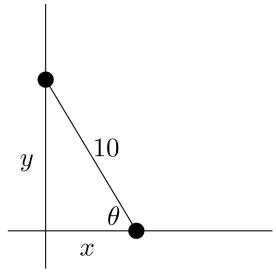
3. **(18 points)** The top of a 10 foot ladder is sliding down a vertical wall at the rate of 1 foot per second. Consider the angle formed by the bottom of the ladder and the ground. How fast is this angle changing when the top of the ladder is 6 feet above the ground?

Solution. Here's the **Picture**:



Variables:
x = distance from bottom of ladder to wall, in ft
y = distance from top of ladder to floor, in ft
θ = angle of ladder with floor, in radians

(And t =time, in sec)

Main **Equation**: $\sin\theta = \dfrac{y}{10}$, i.e., $10\sin\theta = y$

Differentiate (implicitly, w.r.t. time): $10\cos\theta\dfrac{d\theta}{dt} = \dfrac{dy}{dt}$

Use key moment info:

At the key moment, we have $y = 6$ ft and we are told that $\dfrac{dy}{dt} = -1$ft/sec

Also, when $y = 6$, we have $x = 8$, because $x^2 + y^2 = 10^2$, so $x^2 + 36 = 100$, i.e. $x^2 = 64$.

Thus, at the key moment, we have $\cos\theta = \dfrac{\text{adj}}{\text{hyp}} = \dfrac{x}{10} = \dfrac{8}{10}$.

Plugging these values into the derivative equation above,
we have $10 \cdot \left(\dfrac{8}{10}\right)\dfrac{d\theta}{dt} = -1$, so $\dfrac{d\theta}{dt} = -\dfrac{1}{8}$.

That is, the angle is decreasing at $\boxed{\dfrac{1}{8}\text{ rad/sec}}$

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4. **(14 points)** Consider the equation: $xy^3 + y\cos x = 3 + y^2\sin x$
Find the equation of the tangent line to this curve at the point $(0, 3)$.

Solution. Implicit Diff: $y^3 + 3xy^2\dfrac{dy}{dx} + \dfrac{dy}{dx}\cos x - y\sin x = 2y\dfrac{dy}{dx}\sin x + y^2\cos x$,

so at the point $(x, y) = (0, 3)$, we have $3^3 + 0 + \dfrac{dy}{dx}\cos(0) - 3\sin(0) = 2(3)\dfrac{dy}{dx}\sin(0) + 3^2\cos(0)$,

i.e., $27 + \dfrac{dy}{dx} = 9$, i.e., $\dfrac{dy}{dx} = -18$ is the slope of the tangent line.

So the line is $y - 3 = -18(x - 0)$, i.e., $\boxed{y = -18x + 3}$

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5. **(12 points)** Suppose $f(x)$ is a function with the property that

$$f(2) = 5, \quad f'(2) = -1, \quad f(4) = 3, \quad \text{and} \quad f'(4) = 2.$$

Let $g(x) = f(x^2)$ and $h(x) = (f(x))^2$. Compute $g'(2)$ and $h'(2)$.
Solution. $g'(x) = f'(x^2) \cdot 2x$, so $g'(2) = f'(4) \cdot (4) = (2)(4) = \boxed{8}$

$h'(2) = 2(f(2)) \cdot f'(2) = 2(5)(-1) = \boxed{-10}$