Math 111-01, Fall 2022 — Professor Rob Benedetto

Solutions to Practice Problems 2

Differentiation Rules Differentiate the following functions. Simplify your answers unless otherwise specified.

1. $y = \sin^3(x^3)$

Solution. $y' = 3\sin^2(x^3)\cdot\cos(x^3)\cdot 3x^2 = \boxed{9x^2\sin^2(x^3)\cos(x^3)}$

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2. $y = \cos^2(3x)$

Solution. $y' = 2\cos(3x)\cdot(-\sin(3x))\cdot 3 = \boxed{-6\cos(3x)\sin(3x)}$

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3. $f(t) = t^2\sin^5(2t)$

Solution. $f'(t) = t^2[5\sin^4(2t)\cdot\cos(2t)\cdot 2] + \sin^5(2t)\cdot 2t = \boxed{10t^2\sin^4(2t)\cos(2t) + 2t\sin^5(2t)}$

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4. $H(x) = \left(1 - \frac{2}{x^2}\right)^5$

Solution. $H'(x) = 5\left(1 - \frac{2}{x^2}\right)^4\cdot(4x^{-3}) = \boxed{\frac{20}{x^3}\left(1 - \frac{2}{x^2}\right)^4}$

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5. $f(x) = \sqrt[3]{x^3+8}$

Solution. $f'(x) = \frac{1}{3}(x^3+8)^{-2/3}\cdot 3x^2 = \boxed{\frac{x^2}{(x^3+8)^{2/3}}}$

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6. $g(t) = \dfrac{t^3 + \tan\left(\frac{1}{t}\right)}{1+t^2}$

Solution. $$g'(t) = \frac{(1+t^2)\left[3t^2 + \sec^2\left(\frac{1}{t}\right)\cdot\left(-\frac{1}{t^2}\right)\right] - \left(t^3 + \tan\left(\frac{1}{t}\right)\right)\cdot(2t)}{(1+t^2)^2}$$

$$= \boxed{\frac{t^4 + 3t^2 - (1+t^{-2})\sec^2\left(\frac{1}{t}\right) - 2t\tan\left(\frac{1}{t}\right)}{(1+t^2)^2}}$$

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7. $p(x) = \dfrac{1}{(-2x+3)^5}$

Solution. $p(x) = (-2x+3)^{-5}$, so $p'(x) = -5(-2x+3)^{-6}\cdot(-2) = \boxed{\frac{10}{(-2x+3)^{-6}}}$

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8. $r(x) = \dfrac{(2x+1)^3}{(3x+1)^4}$

Solution. $$r'(x) = \frac{(3x+1)^4\cdot 3(2x+2)^2\cdot(2) - (2x+1)^3\cdot 4(3x+1)^3\cdot(3)}{(3x+1)^8}$$

$$= \frac{6(2x+1)^2(3x+1)^3[(3x+1) - 2(2x+1)]}{(3x+1)^8} = \frac{6(2x+1)^2[3x+1-4x-2]}{(3x+1)^5}$$

$$= \frac{6(2x+1)^2(-x-1)}{(3x+1)^5} = \boxed{\frac{-6(x+1)(2x+1)^2}{(3x+1)^5}}$$

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9. $S(x) = \left(\dfrac{1+2x}{1+3x}\right)^4$

Solution. $S'(x) = 4\Big(\dfrac{1+2x}{1+3x}\Big)^3 \cdot \Big[\dfrac{(1+3x)\cdot 2 - (1+2x)\cdot 3}{(1+3x)^2}\Big]$

$= \dfrac{4(1+2x)^3\cdot[2+6x-3-6x]}{(1+3x)^5} = \boxed{\dfrac{-4(1+2x)^3}{(1+3x)^5}}$

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10. $g(x) = \cos(3x)\sin(4x)$

Solution. $g'(x) = \cos(3x)\cos(4x)\cdot 4 + \sin(4x)\big(-\sin(3x)\big)\cdot(3) = \boxed{4\cos(3x)\cos(4x) - 3\sin(3x)\sin(4x)}$

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11. $g(x) = (x+7x^{-6})\sqrt{2x+1}\cos^2(6x)$

Solution. $g(x) = (x+7x^{-6})\big[\sqrt{2x+1}\cos^2(6x)\big]$, so

$g'(x) = (x+7x^{-6})\Big[\sqrt{2x+1}\cdot 2\cos(6x)\cdot\big(-\sin(6x)\big)\cdot(6) + \cos^2(6x)\cdot\dfrac{1}{2\sqrt{2x+1}}\cdot 2\Big]$

$\quad + \big[\sqrt{2x+1}\cos^2(6x)\big](1-42x^{-7})$

$= \boxed{(x+7x^{-6})\cos(6x)\Big[-12\sqrt{2x+1}\sin(6x) + \dfrac{\cos(6x)}{\sqrt{2x+1}}\Big] + \sqrt{2x+1}(1-42x^{-7})\cos^2(6x)}$

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12. $g(x) = \dfrac{\cos(3x)}{\sin(4x)}$

Solution. $g'(x) = \dfrac{\sin(4x)\big(-\sin(3x)\cdot 3\big) - \cos(3x)\cos(4x)\cdot 4}{\sin^2(4x)}$

$= \boxed{\dfrac{-3\sin(3x)\sin(4x) - 4\cos(3x)\cos(4x)}{\sin^2(4x)}}$

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13. $w = \dfrac{5(1+x^2)^3}{x\sqrt{2x+1}}$

Solution. $w' = \dfrac{x\sqrt{2x+1}\big(15(1+x^2)^2\cdot(2x)\big) - 5(1+x^2)^3\big[x\cdot\frac{1}{2\sqrt{2x+1}}\cdot 2 + \sqrt{2x+1}\cdot 1\big]}{x^2(2x+1)}$

$= \dfrac{30x^2(2x+1)(1+x^2)^2 - 5(1+x^2)^3[x+(2x+1)]}{x^2(2x+1)^{3/2}}$

$= \dfrac{5(1+x^2)^2\big(6x^2(2x+1) - (3x+1)(1+x^2)\big)}{x^2(2x+1)^{3/2}}$

$= \dfrac{5(1+x^2)\big(12x^3+6x^2-3x-1-3x^3-x^2\big)}{x^2(2x+1)^{3/2}} = \boxed{\dfrac{5(1+x^2)(9x^3+5x^2-3x-1)}{x^2(2x+1)^{3/2}}}$

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14. $y = \big((x^2+3x)^4 + x\big)^{-5/7}$

Solution. $y' = -\dfrac{5}{7}\big[(x^2+3x)^4+x\big]^{-12/7}[4(x^2+3x)^3(2x+3)+1]$

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15. $g(t) = \cos\left(\sin^3\left(\dfrac{t}{\sqrt{t+1}}\right)\right)$

Solution. $g'(t) = -\sin\left(\sin^3\left(\dfrac{t}{\sqrt{t+1}}\right)\right)\cdot 3\sin^2\left(\dfrac{t}{\sqrt{t+1}}\right)\cdot\cos\left(\dfrac{t}{\sqrt{t+1}}\right)\cdot\Big[\dfrac{\sqrt{t+1}\cdot(1) - t\cdot\frac{1}{2\sqrt{t+1}}\cdot 1}{t+1}\Big]$

Note that $\dfrac{\sqrt{t+1}\cdot(1) - t\cdot\frac{1}{2\sqrt{t+1}}\cdot 1}{t+1} = \dfrac{t+1-\frac{t}{2}}{(t+1)^{3/2}} = \dfrac{t+2}{2(t+1)^{3/2}}$. So

$g'(t) = \dfrac{-3(t+2)}{2(t+1)^{3/2}}\sin\left(\sin^3\left(\dfrac{t}{\sqrt{t+1}}\right)\right)\sin^2\left(\dfrac{t}{\sqrt{t+1}}\right)\cos\left(\dfrac{t}{\sqrt{t+1}}\right)$

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16. $g(x) = \cos^2(6x)\left(\dfrac{\tan x}{\sqrt{2x+1}}\right)$

Solution.

$$g'(x) = \cos^2(6x)\left[\frac{\sqrt{2x+1}\cdot\sec^2 x - \tan x\frac{1}{2\sqrt{2x+1}}(2)}{2x+1}\right] + \left(\frac{\tan x}{\sqrt{2x+1}}\right)\cdot 2\cos(6x)\cdot\big(-\sin(6x)\big)\cdot(6)$$

$$= \cos^2(6x)\left[\frac{(2x+1)\sec^2 x - \tan x}{(2x+1)^{3/2}}\right] + \frac{12\tan x\cos(6x)\sin(6x)}{\sqrt{2x+1}}$$

Differentiation Rules Compute the following derivatives, and simplify your answers. (In some cases you may want to simplify **before** differentiating, but I won't give you that hint on the exam.)

17. $k'(x)$, where $k(x) = \dfrac{1}{x} + 5x^3$.

Solution. $k(x) = x^{-1} + 5x^3$, so $k'(x) = -x^{-2} + 15x^2 = \boxed{-\dfrac{1}{x^2} + 15x^2}$

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18. $H'(x)$, where $H(x) = (x + \sqrt{x^4+1})^5$

Solution. $H'(x) = 5\big(x+\sqrt{x^4+1}\big)^4[1+\frac{1}{2}(x^4+1)^{-1/2}(4x^3)] = \boxed{5\big(x+\sqrt{x^4+1}\big)^4[1+2x^3(x^4+1)^{-1/2}]}$

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19. y', where $\dfrac{x}{y+1} = x^2 - y^2$.

Solution. Differentiating $\dfrac{x}{y+1} = x^2 - y^2$ implicitly gives

$\dfrac{(y+1) - xy'}{(y+1)^2} = 2x - 2yy'$, so $(y+1) - xy' = 2x(y+1)^2 - 2y(y+1)^2y'$, so

$[2y(y+1)^2 - x]y' = 2x(y+1)^2 - (y+1)$, which means $\boxed{y' = \dfrac{2x(y+1)^2 - (y+1)}{2y(y+1)^2 - x}}$

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20. $\dfrac{dv}{dx}$, where $v(x) = \dfrac{x^2+2x-7}{\sqrt[3]{x}}$.

Solution. We have $v(x) = x^{-1/3}(x^2+2x-7) = x^{5/3} + 2x^{2/3} - 7x^{-1/3}$,

so $\dfrac{dv}{dx} = \boxed{\dfrac{5}{3}x^{2/3} + \dfrac{4}{3}x^{-1/3} + \dfrac{7}{3}x^{-4/3}}$

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21. $f'(x)$, where $f(x) = \sqrt[3]{x^2 + \tan x}$

Solution. $f'(x) = \boxed{\dfrac{1}{3}(x^2+\tan x)^{-2/3}(2x + \sec^2 x)}$

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22. $r'(x)$, where $r(x) = x^3\sec^2(5x-3)$

Solution. $r'(x) = 3x^2\sec^2(5x-3) + x^3\cdot 2\sec(5x-3)\sec(5x-3)\tan(5x-3)\cdot 5$

$= \boxed{x^2\big(3 + 10x\tan(5x-3)\big)\sec^2(5x-3)}$

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23. $\dfrac{dh}{dt}$, where $h(t) = (3t-1)^7(4t+3)^9$

Solution. $\dfrac{dh}{dt} = 7(3t-1)^6\cdot 3\cdot(4t+3)^9 + (3t-1)^7\cdot 9(4t+3)^8\cdot 4$

$= 21(3t-1)^6(4t+3)^9 + 36(3t-1)^7(4t+3)^8$ (can stop yet!!!!):

$= 3(3t-1)^6(4t+3)^8[7(4t+3) + 12(3t-1)] = \boxed{3(64t+9)(3t-1)^6(4t+3)^8}$

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24. $\dfrac{dF}{dy}$, where $F(y) = y^2\cos\sqrt{y}$

Solution. $\dfrac{dF}{dy} = 2y\cos\sqrt{y} - y^2\sin\sqrt{y}\cdot\dfrac{1}{2\sqrt{y}} = \boxed{2y\cos\sqrt{y} - \dfrac{1}{2}y^{3/2}\sin\sqrt{y}}$

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25. $\dfrac{dG}{du}$, where $G(u) = \dfrac{3u^2}{u^3 - 5\sin u}$

Solution. $\dfrac{dG}{du} = \dfrac{6u(u^3 - 5\sin u) - 3u^2(3u^2 - 5\cos u)}{(u^3 - 5\sin u)^2} = \boxed{\dfrac{-3u^4 - 30u\sin u + 15u^2\cos u}{(u^3 - 5\sin u)^2}}$

More Derivatives:

26. Find the second derivative of each of the following functions. (Hint: you might save some work by simplifying. And I won't give you that hint on the test.)

$$f(x) = \frac{x^2 + 3x - 7}{\sqrt{x}} \qquad g(t) = \frac{t^5 - t^3}{t^2 - 1} \qquad h(x) = x\sqrt{x^2 - 4}$$

Solution. $f(x) = \dfrac{x^2 + 3x - 7}{\sqrt{x}}$: First, note: $f(x) = \dfrac{x^2 + 3x - 7}{\sqrt{x}} = x^{3/2} + 3x^{1/2} - 7x^{-1/2}$. So $f'(x) = \dfrac{3}{2}x^{1/2} + \dfrac{3}{2}x^{-1/2} + \dfrac{7}{2}x^{-3/2}$, so $\boxed{f''(x) = \dfrac{3}{4}x^{-1/2} - \dfrac{3}{4}x^{-3/2} - \dfrac{21}{4}x^{-5/2}}$

$g(t) = \dfrac{t^5 - t^3}{t^2 - 1}$: First, note: $g(t) = \dfrac{t^5 - t^3}{t^2 - 1} = \dfrac{t^3(t^2 - 1)}{t^2 - 1} = t^3$. So $g'(t) = 3t^2$; so $\boxed{g''(t) = 6t}$

$h(x) = x\sqrt{x^2 - 4}$ has no obvious cancellations. So we compute:

$h'(x) = (x^2 - 4)^{1/2} + \dfrac{x}{2}(x^2 - 4)^{-1/2}(2x) = (x^2 - 4)^{1/2} + x^2(x^2 - 4)^{-1/2} = (x^2 - 4)^{-1/2}(x^2 - 4 + x^2)$

$= (2x^2 - 4)(x^2 - 4)^{-1/2}$.

So $h''(x) = (4x)(x^2 - 4)^{-1/2} + (2x^2 - 4)\left(-\dfrac{1}{2}\right)(x^2 - 4)^{-3/2}(2x)$

$= 4x(x^2 - 4)^{-1/2} - 2x(x^2 - 2)(x^2 - 4)^{-3/2} = 2x(2x^2 - 8 - x^2 + 2)(x^2 - 4)^{-3/2}$

$= \boxed{2x(x^2 - 6)(x^2 - 4)^{-3/2}}$

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27. Let f and g be two differentiable functions, and suppose that their values and the values of their derivatives at $x = 1, 2, 3$ are given by the following table:

x	1	2	3
$f(x)$	3	2	5
$f'(x)$	-2	1	3
$g(x)$	3	1	4
$g'(x)$	-3	2	7

Let $h(x) = f(g(x))$ and $k(x) = f(x) \cdot g(f(x))$. Compute $h'(2)$ and $k'(1)$.

Solution. We have $h'(2) = f'(g(2))g'(2) = f'(1) \cdot 2 = (-2) \cdot 2 = \boxed{-4}$

and $k'(1) = f(1) \cdot g'(f(1)) \cdot f'(1) + g(f(1)) \cdot f'(1) = 3 \cdot g'(3) \cdot (-2) + g(3) \cdot (-2)$

$= 3 \cdot 7 \cdot (-2) + 4 \cdot (-2) = \boxed{-50}$

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28. Let f and g be two differentiable functions, and suppose that their values and the values of their derivatives at $x = 2, 3$ are given by the following table:

x	2	3
$f(x)$	4	0
$f'(x)$	1	-7
$g(x)$	3	-1
$g'(x)$	-5	4

Let $h(x) = f(x)g(x), k(x) = \dfrac{f(x)}{g(x)}$, and $W(x) = f \circ g(x)$. Compute $h'(2)$ and $k'(2)$ and $W'(2)$.

Solution. We have $h'(2) = f(2)g'(2) + f'(2)g(2) = 4 \cdot (-5) + 1 \cdot 3 = \boxed{-17}$

We have $k'(2) = \dfrac{g(2)f'(2) - f(2)g'(2)}{(g(2))^2} = \dfrac{3(1) - 4(-5)}{9} = \boxed{\dfrac{23}{9}}$

We have $W'(2) = f'(g(2)) \cdot g'(2) = f'(3) \cdot (-5) = (-7)(-5) = \boxed{35}$

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29. Let $f(x) = \sqrt{x+1} \cdot g(x)$ where $g(0) = -7$ and $g'(0) = 4$. Compute $f'(0)$.

Solution. We have $f'(x) = \sqrt{x+1} \cdot g'(x) + g(x)\dfrac{1}{2\sqrt{x+1}}$.

So $f'(0) = \sqrt{0+1} \cdot g'(0) + g(0) \cdot \dfrac{1}{2\sqrt{0+1}} = 1 \cdot 4 + (-7) \cdot \dfrac{1}{2} = \boxed{\dfrac{1}{2}}$

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30. Let $f(x) = \dfrac{\sqrt{x^2+1}}{g(x)}$ where $g(0) = -7$ and $g'(0) = 4$. Compute $f'(0)$.

Solution. We have $f'(x) = \dfrac{g(x)\dfrac{1}{2\sqrt{x^2+1}}(2x) - \sqrt{x^2+1}g'(x)}{(g(x))^2}$.

So $f'(0) = \dfrac{g(0)\dfrac{1}{2\sqrt{1}}(0) - \sqrt{1}g'(0)}{(g(0))^2} = \boxed{-\dfrac{4}{49}}$

More Implicit Differentiation: For each of the equations described below, find an equation of the tangent line to the curve at the given point.

31. $x^3 + x^2y + 4y^2 = 6$ at $(1, 1)$.

Solution. Differentiating $x^3 + x^2y + 4y^2 = 6$ implicitly gives $3x^2 + 2xy + x^2\dfrac{dy}{dx} + 8y\dfrac{dy}{dx} = 0$, so $(x^2 + 8y)\dfrac{dy}{dx} = -3x^2 - 2xy$, so $\dfrac{dy}{dx} = \dfrac{-3x^2 - 2xy}{x^2 + 8y}$.

At $(1, 1)$, the slope of the tangent line is therefore $\dfrac{dy}{dx} = (-3-2)/(1+8) = -5/9$.

The point-slope formula gives $y - 1 = -\dfrac{5}{9}(x-1)$; that is, $\boxed{y = -\dfrac{5}{9}x + \dfrac{14}{9}}$

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32. $4(x+y)^2 = x^2y^2$ at $(-2, 1)$.

Solution. Differentiating $4(x+y)^2 = x^2y^2$ implicitly gives $8(x+y)(1 + \dfrac{dy}{dx}) = 2xy^2 + 2x^2y\dfrac{dy}{dx}$, so $4x + 4y + \dfrac{dy}{dx}(4x+4y) = xy^2 + x^2y\dfrac{dy}{dx}$, so $\dfrac{dy}{dx}(4x + 4y - x^2y) = xy^2 - 4x - 4y$, so $\dfrac{dy}{dx} = \dfrac{xy^2 - 4x - 4y}{4x + 4y - x^2y}$.

At $(-2, 1)$, the slope of the tangent line is therefore $\dfrac{dy}{dx} = (-2+8-4)/(-8+4-4) = -1/4$.

By the point-slope formula, $y - 1 = -\dfrac{1}{4}(x+2)$; that is, $\boxed{y = -\dfrac{1}{4}x + \dfrac{1}{2}}$

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33. $\dfrac{x}{y+1} = x^2 - y^2$ at $(1, 0)$.

Solution. Differentiating $\dfrac{x}{y+1} = x^2 - y^2$ implicitly gives $\dfrac{(y+1) - x\dfrac{dy}{dx}}{(y+1)^2} = 2x - 2y\dfrac{dy}{dx}$, so $y + 1 - x\dfrac{dy}{dx} = 2x(y+1)^2 - 2y(y+1)^2\dfrac{dy}{dx}$, so $(2y(y+1)^2 - x)\dfrac{dy}{dx} = 2x(y+1)^2 - y - 1$, so $\dfrac{dy}{dx} = \dfrac{2x(y+1)^2 - y - 1}{2y(y+1)^2 - x}$.

At $(1,0)$, the slope of the tangent line is therefore $\dfrac{dy}{dx} = (2-1)/(-1) = -1$.

The point-slope formula gives $y - 0 = -1(x-1)$; that is, $\boxed{y = -x + 1}$

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34. $4\cos x \sin y = 3$ at $(\pi/6, \pi/3)$.

Solution. Differentiating gives: $-4\sin x \sin y + 4(\cos x \cos y)\dfrac{dy}{dx} = 0$, and so $\dfrac{dy}{dx} = \dfrac{\sin x \sin y}{\cos x \cos y}$.

At $(\pi/6, \pi/3)$, this means $\dfrac{dy}{dx} = \dfrac{(1/2)(\sqrt{3}/2)}{(\sqrt{3}/2)(1/2)} = 1$.

So the tangent line is $y - \pi/3 = x - \pi/6$, that is, $\boxed{y = x + \pi/6}$

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35. $y^3 - xy^2 + \cos(xy) = 2$ at $(0,1)$

Solution. Differentiating gives: $3y^2\dfrac{dy}{dx} - \left(2xy\dfrac{dy}{dx} + y^2\right) - \sin(xy)\left(x\dfrac{dy}{dx} + y\right) = 0$,

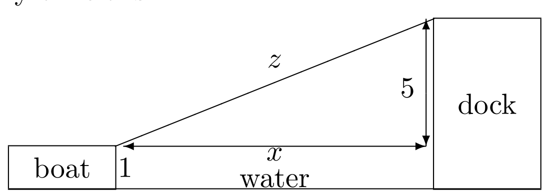
and at $(0,1)$, this means $3\dfrac{dy}{dx} - \left(0\dfrac{dy}{dx} + 1\right) - \sin(0)\left(0\dfrac{dy}{dx} + 1\right) = 0$ which implies $\dfrac{dy}{dx} = \dfrac{1}{3}$.

So the tangent line is $y - 1 = \frac{1}{3}(x - 0)$, that is, $\boxed{y = \dfrac{1}{3}x + 1}$

Related Rates:

36. A rowboat is in the water near a dock, and a rope attached to the bow of the boat is connected to a winch on the dock. The winch is 6 ft above the water, and the bow of the boat is 1 ft above the water. The winch pulls the boat towards the dock by retracting 1 ft of rope per second, keeping the rope taut. When the boat is 12 feet from the dock, how fast is it moving towards the dock?

Solution. Picture at arbitrary time t is:



Variables:
x = horizontal distance of boat to dock, in feet
z = length of rope, in feet

Equation: $x^2 + 25 = z^2$.

Differentiate both sides w.r.t. time t: $2x\dfrac{dx}{dt} = 2z\dfrac{dz}{dt}$

Solve: at key moment, $\dfrac{dz}{dt} = -1$ and $x = 12$.

So in original equation, $z = \sqrt{144 + 25} = 13$.

So at that moment, $\dfrac{dx}{dt} = \dfrac{z}{x}\dfrac{dz}{dt} = -\dfrac{13}{12}$.

So the boat is moving towards the dock at $\boxed{\dfrac{13}{12} \text{ feet per second}}$

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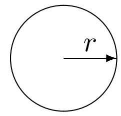
37. A hot circular plate of metal is cooling. As it cools its radius is decreasing at the rate of 0.01 cm/min. At what rate is the plate's area decreasing when the radius equals 50 cm?

Solution. Diagram:



Variables:
r = radius of the metal plate at time t, in cm
A = area of the metal plate at time t, in cm^2
Find $\frac{dA}{dt}$ =? when $r = 50$ cm and $\frac{dr}{dt} = -0.01\frac{\text{cm}}{\text{min}}$
Equation: $A = \pi r^2$.
Differentiate both sides w.r.t. time t: $\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$
Solve: at key moment, $\frac{dA}{dt} = 2\pi(50)(-0.01)$
So: $\frac{dA}{dt} = 100\pi(-0.01) = -\pi\frac{\text{cm}^2}{\text{min}}$
Answer: The plate's area is shrinking at a rate of $\boxed{\pi \text{ cm}^2/\text{min}}$

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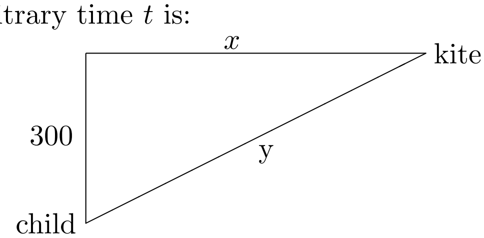
38. A child is flying a kite at a height of 300 feet above the level of the child's hand. The kite is being blown by the wind so that it is moving horizontally at 25 ft/sec. Assuming that the string is always a straight line from the child's hand to the kite, how fast is the child letting out string at the moment when the kite is 500 ft from the child?

Solution. Picture at arbitrary time t is:



Variables:
x = distance height is horizontally from above the child's head, in feet
y = length of string, in feet

Equation: $x^2 + 300^2 = y^2$

Differentiate w.r.t. time t: $2x\frac{dx}{dt} = 2y\frac{dy}{dt}$

Solve: At the key instant, $y = 500$ because the kite is 500ft from the child.
Ny the original equation $x^2 + 300^2 = y^2$, we get $x^2 = 500^2 - 300^2 = (100)^2(25-9) = 16(100)^2 = 400^2$, so $x = 400$ft.
Since $\frac{dx}{dt} = 25$ft/sec, we get

$$2(400)(25) = 2(500)y', \quad \text{so} \quad \frac{dy}{dt} = \frac{20000}{1000} = 20.$$

So the child must be letting out kite string at $\boxed{20\text{ft/sec}}$

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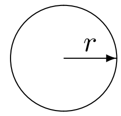
39. Suppose a snowball remains spherical while it melts, with the radius shrinking at one inch per hour. How fast is the volume of the snowball decreasing when the radius is 2 inches?

Solution. Diagram (cross-sectioned in 2 dimensions here):



Variables:
r = radius of the sphere at time t, in inches
V = volume of the sphere at time t, in in^3
Find $\frac{dV}{dt}$ =? when $r = 2$ feet and $\frac{dr}{dt} = -1\frac{\text{in}}{\text{hr}}$

Equation: $V = \frac{4}{3}\pi r^3$

Differentiate w.r.t. time t: $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

Solve: with $\frac{dr}{dt} = -1$:
$\frac{dV}{dt} = 4\pi(2)^2(-1) = -16\pi\frac{\text{in}^3}{\text{hr}}$

So: The volume of the snowball is decreasing at a rate of $\boxed{16\pi \text{ cubic inches per hour}}$

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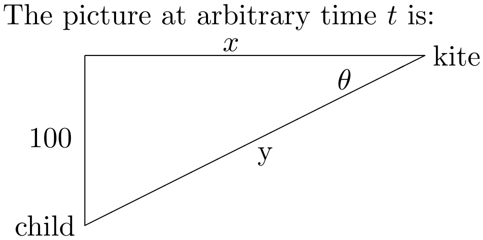
40. A kite 100 feet high is being blown horizontally at 8 feet per second. When there are 300 feet of string out, (a) how fast is the string running out? (b) how fast is the angle between the string and the horizontal changing?

Solution: (a): Diagram: The picture at arbitrary time t is:



Variables:
x = distance kite has travelled horizontally at time t, in feet
y = distance between kite and child at time t, in feet
Find $\frac{dy}{dt}$ =? when $y = 300$ feet and $\frac{dx}{dt} = 8\frac{\text{ft}}{\text{sec}}$

Equation: $x^2 + 100^2 = y^2$

Differentiate w.r.t. time t: $2x\frac{dx}{dt} = 2y\frac{dy}{dt}$, so $x\frac{dx}{dt} = y\frac{dy}{dt}$

Solve:
when $y = 300$, we have $x = \sqrt{(300)^2 - (100)^2} = \sqrt{80000} = 200\sqrt{2}$. So, $200\sqrt{2} \cdot 8 = 300\frac{dy}{dt}$

So: $\frac{dy}{dt} = \frac{1600\sqrt{2}}{300} = \frac{16\sqrt{2}}{3}\frac{\text{ft}}{\text{sec}}$

Answer: The string is running out at a rate of $\boxed{\frac{16\sqrt{2}}{3} \text{ feet per second}}$

(b): Same diagram, with one new variable (which I already put in the diagram above):
θ = the angle between the string/horizontal, in radians

Equation: $\tan\theta = \frac{100}{x}$.

Differentiate: $\sec^2\theta \frac{d\theta}{dt} = -\frac{100}{x^2}\frac{dx}{dt}$, so $\frac{d\theta}{dt} = -\frac{100}{x^2}\cdot\cos^2\theta \frac{dx}{dt}$.

Solve: At the key instant when $y = 300$, using the original equation, we have $x = \sqrt{(300)^2 - (100)^2} = \sqrt{80000} = 200\sqrt{2}$.

Therefore, $\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{200\sqrt{2}}{300} = \frac{2\sqrt{2}}{3}$.

So $\frac{d\theta}{dt} = -\frac{100}{(200\sqrt{2})^2}\cdot\left(\frac{2\sqrt{2}}{3}\right)^2\cdot 8 = -\frac{1}{800}\cdot\frac{8}{9}\cdot 8 = -\frac{8}{900}$

So the angle is decreasing at a rate of $\boxed{\frac{8}{900} \text{ radians per second}}$

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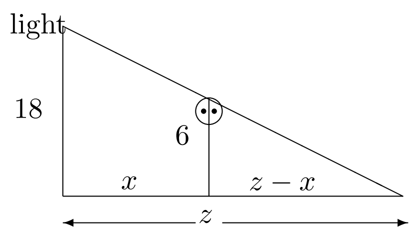
41. A 6 foot tall man walks with a speed of 8 feet per second away from a street light that is atop an 18 foot pole. How fast is the top of his shadow moving along the ground when he is 100 feet from the light pole?

Solution. Diagram:



Variables:
x = man's distance from pole at time t, in ft
z = distance from tip of shadow to pole at time t, in ft
$\frac{dz}{dt} =?$ when $x = 100$ feet and $\frac{dx}{dt} = 8\frac{\text{ft}}{\text{sec}}$ (He's fast!)

Equation (via similar triangles): $\frac{z}{18} = \frac{z-x}{6}$, i.e., $6z = 18z - 18x$, i.e., $3x = 2z$.

Differentiate w.r.t. time t: $3\frac{dx}{dt} = 2\frac{dz}{dt}$

Solve: This means $3(8) = 2\frac{dz}{dt}$, i.e., $\frac{dz}{dt} = 12\frac{\text{ft}}{\text{sec}}$

So: The tip of his shadow is moving along the ground at a rate of $\boxed{\text{12 feet per second}}$

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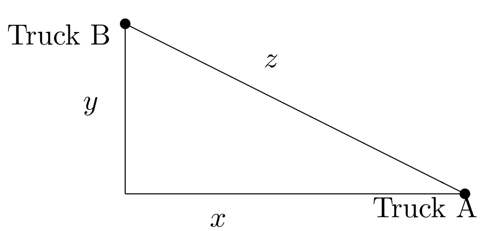
42. Two trucks leave a depot at the same time. Truck A travels east at 40 miles per hour, while Truck B travels north at 30 miles per hour. How fast is the distance between the trucks changing 60 minutes after leaving the depot?

Solution. Diagram: here's the Picture:



Variables:
x = distance Truck A travelled East at time t, in miles

$y =$ distance Truck B travelled North at time t, in miles
$z =$ distance between Trucks A and B at time t, in miles
Find $\frac{dz}{dt} =?$ after 1 hour, when $x = 40$ miles, $y = 30$, $\frac{dx}{dt} = 40$ m.p.h. and $\frac{dy}{dt} = 30$ m.p.h.

Equation: $x^2 + y^2 = z^2$

Differentiate both sides w.r.t. time t: $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$, i.e., $x\frac{dx}{dt} + y\frac{dy}{dt} = z\frac{dz}{dt}$,

Solve:
At key moment, original equation gives $z = \sqrt{(40)^2 + (30)^2} = 50$,
so derivative equation is: $40(40) + 30(30) = 50\frac{dz}{dt}$

so $\frac{dz}{dt} = \frac{1600 + 900}{50} = 50$ m.p.h.

Answer: The distance between the trucks is increasing at a rate of $\boxed{\text{50 miles per hour}}$

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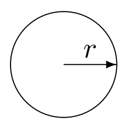
43. Suppose a spherical balloon is inflated at the rate of 10 cubic inches per minute. How fast is the radius of the balloon increasing when the radius is 5 inches?

Solution. Diagram (cross-sectioned in 2 dimensions here):



Variables:
$r =$ radius of the sphere at time t, in inches
$V =$ volume of the sphere at time t, in in^3
Find $\frac{dr}{dt} =?$ when $r = 5$in, and $\frac{dV}{dt} = 10\frac{\text{in}^3}{\text{min}}$

Equation: $V = \frac{4}{3}\pi r^3$

Differentiate w.r.t. time t: $\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$

Solve: $10 = 4\pi(5)^2\frac{dr}{dt}$,
so $\frac{dr}{dt} = \frac{10}{100\pi} = \frac{1}{10\pi}\frac{\text{in}}{\text{min}}$

So: The radius of the balloon is increasing at a rate of $\boxed{\frac{1}{10\pi}\text{ inches per minute}}$

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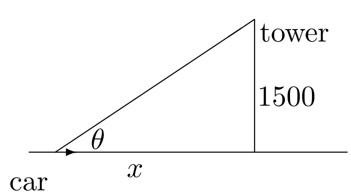
44. A child riding in a car driving along a straight road is looking through binoculars when she sees a water tower off to the side. The tower is located 1500 ft from the nearest point on the road. At a particular moment, the car is moving at 80 feet per second, and the car is 800 feet from that nearest point to the tower. How fast must the child be rotating the angle that the binoculars are pointing to keep the tower in view?

Solution. Diagram:



Variables:
x =distance from car to nearest point to tower at time t, in ft
θ = angle binoculars form with straight road at time t, in radians
Find $\frac{d\theta}{dt}$ =? when $x = 800$ feet and $\frac{dx}{dt} = -80\frac{\text{ft}}{\text{sec}}$
(Note this derivative is negative because in the picture, we have the car driving left to right towards the nearest point.)

Equation: $\tan\theta = \frac{1500}{x}$

Differentiate w.r.t. time t: $\sec^2\theta\frac{d\theta}{dt} = -\frac{1500}{x^2}\frac{dx}{dt}$

Solve: At the key moment, the hypoteneuse is
$\sqrt{x^2 + 1500^2} = \sqrt{800^2 + 1500^2} = 100\sqrt{8^2 + 15^2} = 100\sqrt{289} = 1700$

so at that moment, $\sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{1700}{800} = \frac{17}{8}$.

So: $\left(\frac{17}{8}\right)^2\frac{d\theta}{dt} = -\frac{1500}{(800)^2}(-80)$. That is, $\frac{d\theta}{dt} = \frac{-\frac{1500}{800^2}\cdot(-80)}{(17/8)^2} = \frac{12}{289}$ radians/sec.

So: The child must be rotating the binoculars at a rate of $\boxed{\frac{12}{289}\text{ radians per second}}$

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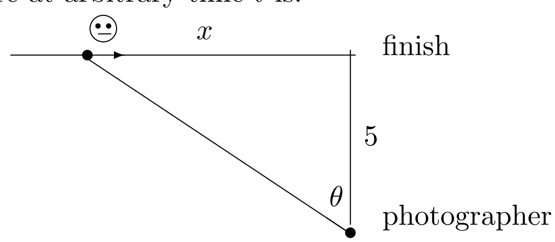
45. A photographer is televising a 100-meter dash from a position 5 meters from the track in line with the finish line. When the runners are 12 meters from the finish line, the camera is turning at the rate of $\frac{3}{5}$ radians per second. How fast are the runners moving then?

Solution. Diagram: The picture at arbitrary time t is:



Variables:
x = distance between runners and finish line, in meters
θ = angle camera is turned from finish line, in radians

Find $\frac{dx}{dt}$ =? when $x = 12$ and $\frac{d\theta}{dt} = -\frac{3}{5}\frac{\text{rad}}{\text{sec}}$

Equation: $\tan\theta = \frac{x}{5}$.

Differentiate:
$\sec^2\theta\frac{d\theta}{dt} = \frac{1}{5}\cdot\frac{dx}{dt}$, i.e., $\frac{dx}{dt} = 5\sec^2\theta\frac{d\theta}{dt}$

Solve: By the Pythagorean Theorem, hyp $= \sqrt{5^2 + (12)^2} = \sqrt{169} = 13$.

So $\sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{13}{5}$

So with $\frac{d\theta}{dt} = \frac{3}{5}$, we get $\frac{dx}{dt} = 5\left(\frac{13}{5}\right)^2\cdot\frac{3}{5} = \frac{3\cdot 13^2}{5^2} = \frac{507}{25}$.

So the runners are moving at $\boxed{\frac{507}{25} \text{ meters per second}}$

Absolute Extrema: Find the absolute maximum and absolute minimum values of the following functions on the given intervals.

46. $f(x) = 3x^{2/3} - \frac{x}{4}$ on $[-1, 1]$.

Solution. (By Closed Interval Method):

$f'(x) = 2x^{-1/3} - \frac{1}{4}$, which is defined everywhere **except** $x = 0$

Solving $f' = 0$ gives $2x^{-1/3} = \frac{1}{4}$, i.e., $8 = x^{1/3}$, i.e., $x = 8^3 = 512$, which is not in the interval

So the only critical number is $x = 0$. Testing it and endpoints:

$$f(-1) = 3(1) + \frac{1}{4} = \frac{13}{4}, \quad f(0) = 0 - 0 = 0, \quad f(1) = 3 - \frac{1}{4} = \frac{11}{4}.$$

So the $\boxed{\text{absolute maximum is } \frac{13}{4}}$ and the $\boxed{\text{absolute minimum is } 0}$

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47. $h(x) = \frac{x^2 - 1}{x^2 + 1}$ on $[-1, 3]$.

Solution. (By Closed Interval Method):

$h'(x) = \frac{2x(x^2+1) - 2x(x^2-1)}{(x^2+1)^2} = \frac{2x^3 + 2x - 2x^3 + 2x}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2}$,
which is **always defined** (because the denominator is never zero).

Solving $h' = 0$ gives $x = 0$.

So the only critical number is $x = 0$. Testing it and endpoints:

$$h(-1) = 0, \quad h(0) = \frac{-1}{1} = -1, \quad h(3) = \frac{8}{10} = \frac{4}{5}.$$

So the $\boxed{\text{absolute maximum is } \frac{4}{5}}$ and the $\boxed{\text{absolute minimum is } -1}$

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48. $F(x) = -2x^3 + 3x^2$ on $[-\frac{1}{2}, 3]$.

Solution. (By Closed Interval Method):

$F'(x) = -6x^2 + 6x = -6x(x-1)$
which is **always defined**.

Solving $F' = 0$ gives $x = 0, 1$, both of which are in the interval. Testing them and endpoints:

$F\left(\frac{-1}{2}\right) = -2\left(-\frac{1}{8}\right) + 3\left(\frac{1}{4}\right) = 1, \quad F(0) = 0, \quad F(1) = -2 + 3 = 1, \quad F(3) = -2(27) + 3(9) = -27$

So the $\boxed{\text{absolute maximum is } 1}$ and the $\boxed{\text{absolute minimum is } -27}$

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49. $f(x) = x^{2/3}$ on $[-1, 8]$.

Solution. (By Closed Interval Method):

$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$, which is defined everywhere **except** $x = -0$.

Solving $f' = 0$ gives no solutions. So the only critical number is $x = 0$. Testing it and endpoints:

$$f(-1) = \left(\sqrt[3]{-1}\right)^2 = (-1)^2 = 1, \quad f(0) = \left(\sqrt[3]{0}\right)^2 = 0, \quad f(8) = \left(\sqrt[3]{8}\right)^2 = (2)^2 = 4.$$

So the $\boxed{\text{absolute maximum is } 4}$ and the $\boxed{\text{absolute minimum is } 0}$