1. (20 points) You need to construct a box with a square base with a fixed volume of 24 cubic feet. The material for the bottom and top costs $3 per square foot, and the material for the sides costs $1 per square foot. What are the dimensions that minimize the cost required to build such a box? What is that minimum cost?

Solution. Here’s the diagram:

![Diagram of a box with dimensions x and y]

The area of the base is $x^2$, so the base costs $3x^2.
Similarly, the top also costs $3x^2$. So the base and top together cost $6x^2$.
Each of the four sides has area $xy$, and so costs $xy$. Together, the four sides cost $4xy$.

So the total cost of the box (in dollars) is $6x^2 + 4xy$

The volume of the box is $x^2y$, which we set equal to 24. Solving for $y$ gives $y = 24x^{-2}$.

So the cost of the box is $C(x) = 6x^2 + 96x^{-1}$.

The common sense bounds say $x > 0$ [note that $x = 0$ is impossible because $x^2y = 24$]. [And $y > 0$ then gives $24/x^2 > 0$, which tells us nothing new.]

So we must minimize $C(x) = 6x^2 + 96x^{-1}$ on the domain $(0, \infty)$.

Differentiate: $C'(x) = 12x - 96x^{-2}$, which is defined everywhere on $(0, \infty)$.

Solving $C' = 0$ gives $12x^3 = 96$, so $x^3 = 8$, so $x = 2$ is the only critical number.

Using FDTAE, since $C'(x) = 12x^{-2}(x^3 - 8)$, our $C''$ chart is:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$(0, 2)$</th>
<th>$(2, \infty)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C''(x)$</td>
<td>$-$</td>
<td>$+$</td>
</tr>
<tr>
<td>$C(x)$</td>
<td>$\nearrow$</td>
<td>$\nearrow$</td>
</tr>
</tbody>
</table>

So by FDTAE, $C$ has an absolute minimum at $x = 2$ ft.

That gives $y = 24/2^2 = 6$ ft and $C(2) = 6(4) + 96/2 = 72$.

So the best box is $2\text{ ft} \times 2\text{ ft} \times 3\text{ ft}$ and costs $72$.

2. (25 points) Compute $\int_1^3 x^2 - 3x \, dx$ using each of the following two different methods:

(a) The Fundamental Theorem of Calculus.

(b) Riemann Sums and the limit definition of the definite integral.

Solution. (a): Let $f(x) = x^2 - 3x$, and chop the interval $[1, 3]$ into $n$ equal-length intervals.

We have $\Delta x = \frac{3 - 1}{n} = \frac{2}{n}$, and so $x_i = 1 + \frac{2i}{n}$.

The $n$-th right-hand Riemann sum is therefore $R_n = \sum_{i=1}^{n} f(x_i) \Delta x = \sum_{i=1}^{n} f\left(1 + \frac{2i}{n}\right) \cdot \frac{2}{n} = \sum_{i=1}^{n} \left[\left(1 + \frac{2i}{n}\right)^2 - 3\left(1 + \frac{2i}{n}\right)\right] \frac{2}{n}$.
\[
\sum_{i=1}^{n} \left[ 1 + \frac{4i}{n} + \frac{4i^2}{n^2} - 3 - \frac{6i}{n} \right] \frac{2}{n} = \sum_{i=1}^{n} \left[ -2 - \frac{2i}{n} + \frac{4i^2}{n^2} \right] \frac{2}{n}
\]
\[
= -4 \sum_{i=1}^{n} \frac{1}{n} \sum_{i=1}^{n} i - \frac{4}{n^2} \sum_{i=1}^{n} i^2 = \frac{-4}{n} \cdot n \cdot \frac{n+1}{2} + \frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}
\]
\[
= -4 - 2 \left( 1 + \frac{1}{n} \right) + \frac{4}{3} \left( 1 + \frac{1}{n} \right) \left( 2 + \frac{1}{n} \right)
\]

Thus, \( \int_{1}^{3} x^2 - 3x \, dx = \lim_{n \to \infty} R_n = -4 - 2(1) + \frac{4}{3}(1)(2) = -6 + \frac{8}{3} = \frac{-10}{3} \)

(b): \( \int_{1}^{3} x^2 - 3x \, dx = \frac{1}{3} x^3 - \frac{3}{2} x^2 \bigg|_{1}^{3} = \left( \frac{1}{3} \cdot 3^3 - \frac{3}{2} \cdot 3^2 \right) - \left( \frac{1}{3} \cdot 1^3 - \frac{3}{2} \cdot 1^2 \right) = 9 - \frac{27}{2} - \frac{1}{3} + \frac{3}{2} = -3 - \frac{1}{3} = \frac{-10}{3} \)

3. (30 points) Let \( f(x) = \frac{-x^2 + x + 2}{x^2 - 2x + 1} = \frac{-x^2 + x + 2}{(x-1)^2} \).

For this function, discuss domain, vertical and horizontal asymptote(s), interval(s) of increase or decrease, local extreme value(s), concavity, and inflection point(s). Then use this information to present a detailed and labelled sketch of the curve \( y = f(x) \).

You may take my word for it that:

\[
f'(x) = \frac{x - 5}{(x - 1)^3} \quad \text{and} \quad f''(x) = \frac{-2x + 14}{(x - 1)^4}
\]

Solution.

- Domain: \( f(x) \) has domain \( \{x | x \neq 1\} \)
- VA: Vertical asymptote at \( x = 1 \) (because denom = 0 there, and nowhere else).
- HA: Horizontal asymptote is \( y = -1 \) on both sides since \( \lim_{x \to \pm \infty} f(x) = -1 \) because
  \[
  \lim_{x \to \pm \infty} \frac{-x^2 + x + 2}{x^2 - 2x + 1} \cdot \frac{\left( \frac{1}{x^2} \right)}{\left( \frac{1}{x^2} \right)} = \lim_{x \to \pm \infty} \frac{-1 + \frac{1}{x} + \frac{2}{x^2}}{1 - \frac{2}{x} + \frac{1}{x^2}} = -1
  \]
- First Derivative Information:
  We know \( f'(x) = \frac{x - 5}{(x - 1)^3} \). The critical points occur where \( f' \) is undefined or zero. The former happens when \( x = 1 \), but \( x = 1 \) was not in the domain of the original function, so it isn’t technically a critical number. The latter happens when \( x = 5 \). As a result, \( x = 5 \) is the critical number. Our chart for \( f' \) is:

<table>
<thead>
<tr>
<th>( x )</th>
<th>((-\infty, 1))</th>
<th>((1, 5))</th>
<th>((5, \infty))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f'(x) )</td>
<td>+</td>
<td>−</td>
<td>+</td>
</tr>
<tr>
<td>( f(x) )</td>
<td>↗</td>
<td>↘</td>
<td>↗</td>
</tr>
</tbody>
</table>

So \( f \) is decreasing on \((1, 5)\) and increasing on \((-\infty, 1)\) and \((5, \infty)\)
with a local minimum at \( x = 5 \)
(Remember that \( x = 1 \) is a vertical asymptote, not a max.)

- Second Derivative Information:
  Meanwhile, \( f'' = \frac{-2x + 14}{(x - 1)^4} \), which is always defined away from the asymptote \( x = 1 \).
We have \( f'' = 0 \) when \( x = 7 \). Thus, our \( f'' \) chart is

<table>
<thead>
<tr>
<th>( x )</th>
<th>( (-\infty, 1) )</th>
<th>( (1, 7) )</th>
<th>( (7, \infty) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f''(x) )</td>
<td>+</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>( f(x) )</td>
<td>U</td>
<td>U</td>
<td>U</td>
</tr>
</tbody>
</table>

So \( f \) is concave down on \( (7, \infty) \) and concave up on \( (-\infty, 1) \) and \( (1, 7) \) with an inflection point at \( x = 7 \).

[For graph, see separate PDF.]

4. (10 points) Compute the following limits.

(a) \( \lim_{x \to \infty} \frac{x^2 - x + 5}{3x^7 + x^6 - 2022} \)

Solution. \( \lim_{x \to \infty} \frac{x^2 - x + 5}{3x^7 + x^6 - 2022} \cdot \frac{1}{x^7} = \lim_{x \to \infty} \frac{x^5 - x^6 + 5x^{-7}}{3 + x^{-1} - 2022x^{-7}} = 0 \)

(b) \( \lim_{x \to -\infty} \frac{\sqrt{4x^2 + 5x}}{6x + 7} \)

\( \lim_{x \to -\infty} \frac{\sqrt{4x^2 + 5x}}{6x + 7} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x \to -\infty} \frac{\sqrt{4x^2 + 5x}}{6 + 7x^{-1}} = \lim_{x \to -\infty} \frac{-4 + 5x^{-1}}{6 + 7x^{-1}} = \frac{-4}{6} = \frac{-2}{3} \)

5. (15 points) Compute the following definite and indefinite integrals.

(a) \( \int \frac{y^2 - 4y + 1}{\sqrt{y}} \, dy \)

\( \int \frac{y^2 - 4y + 1}{\sqrt{y}} \, dy = \int y^{3/2} - 4y^{1/2} + y^{-1/2} \, dy = \frac{2}{5}y^{5/2} - \frac{8}{3}y^{3/2} + 2y^{1/2} + C \)

(b) \( \int_{-\pi/4}^{5\pi/6} 3x + \cos x \, dx \)

\( \int_{-\pi/4}^{5\pi/6} 3x + \cos x \, dx = \left[ \frac{3\pi^2}{2} + \sin x \right]_{-\pi/4}^{5\pi/6} = \left( \frac{3\pi^2}{2} + \sin \left( \frac{5\pi}{6} \right) \right) - \left( \frac{3\pi}{2} + \sin \left( -\frac{\pi}{4} \right) \right) = \frac{3\pi^2}{2} + \frac{1}{2} - \frac{3\pi^2}{2} - \left( -\frac{\sqrt{2}}{2} \right) = \frac{25}{24} - \frac{3}{32} \pi^2 + \frac{1 + \sqrt{2}}{2} \)