Solutions to Practice Test B for Midterm Exam 3

1. **(10 points)** Find a function $f(x)$ such that $f(1) = 3$, $f'(1) = 5$, and $f''(x) = 12x^2 + 12x$.

Solution. Antidifferentiating $f''(x) = 12x^2 + 12x$ gives $f'(x) = 4x^3 + 6x^2 + C$, for some constant C.

But $5 = f'(1) = 4 + 6 + C$, and hence $C = -5$.

That is, $f'(x) = 4x^3 + 6x^2 - 5$.

Antidifferentiating again, $f(x) = x^4 + 2x^3 - 5x + K$, for some constant K.

But $3 = f(1) = 1 + 2 - 5 + K$, and hence $K = 5$. Thus, $\boxed{f(x) = x^4 + 2x^3 - 5x + 5}$

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2. **(25 points)** Let $f(x) = \dfrac{2x^3 + 45x^2 + 315x + 600}{x^3}$. Take my word for it that

$$f'(x) = \frac{-45(x+4)(x+10)}{x^4}, \qquad \text{and} \qquad f''(x) = \frac{90(x+5)(x+16)}{x^5}.$$

Sketch the graph of $y = f(x)$, clearly indicating **horizontal and vertical asymptotes**, **local extrema**, **inflection points**, and **intervals of increase and decrease and of concavity**.

You do **not** need to indicate locations of intercepts or y-coordinates of extrema or inflection points.

Solution. The vertical asymptotes occur when we divide by zero, i.e., at $\boxed{x = 0}$ [f is defined and continuous everywhere else.]

For the horizontal asymptotes, we compute

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} 2 + \dfrac{45}{x} + \dfrac{315}{x^2} + \dfrac{600}{x^3} = 2$, and $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} 2 + \dfrac{45}{x} + \dfrac{315}{x^2} + \dfrac{600}{x^3} = 2.$

So $\boxed{y = 2 \text{ is a horizontal asymptote on both sides}}$

Meanwhile, f' is defined everywhere except at the vertical asymptote $x = 0$. Moreover, $f'(x) = 0$ exactly at $x = -4$ and $x = -10$, so these are the only critical points. Our f' chart is:

x	$(-\infty, -10)$	$(-10, -4)$	$(-4, 0)$	$(0, \infty)$
$f'(x)$	$-$	$+$	$-$	$-$
$f(x)$	$\searrow$	$\nearrow$	$\searrow$	$\searrow$

There is a $\boxed{\text{local minimum at } x = -10 \text{ and a local maximum at } x = -4}$

Turning to the second derivative, we see that f'' is also defined everywhere except the asymptote $x = 0$; it is zero exactly at $x = -16$ and $x = -5$. The f'' chart is:

x	$(-\infty, -16)$	$(-16, -5)$	$(-5, 0)$	$(0, \infty)$
$f'(x)$	$-$	$+$	$-$	$+$
$f(x)$	$\bigcap$	$\bigcup$	$\bigcap$	$\bigcup$

There are $\boxed{\text{inflection points at } x = -16 \text{ and } x = -5}$ (Note that $x = 0$ is an asymptote, not an inflection point.)

See separate PDF for a sketch of the graph.

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3. **(15 points)** Let $g(x) = 4x^5 - 5x^4 - 40x^3$. Find all critical points of g in $(-\infty, \infty)$, and classify each as a local maximum, local minimum, or neither.

Solution. We compute $g'(x) = 20x^4 - 20x^3 - 120x^2 = 20x^2(x^2 - x - 6) = 20x^2(x-3)(x+2)$, which is **always defined**.

Setting $g' = 0$ gives $x = -2, 0, 3$ as the critical points. Our g' chart is then

x	$(-\infty, -2)$	$(-2, 0)$	$(0, 3)$	$(3, \infty)$
$g'(x)$	$+$	$-$	$-$	$+$
$g(x)$	$\nearrow$	$\searrow$	$\searrow$	$\nearrow$

Thus, by the First Derivative Test, g has a

$\boxed{\text{local maximum at } x = -2, \text{ a local minimum at } x = 3, \text{ and neither at } x = 0}$

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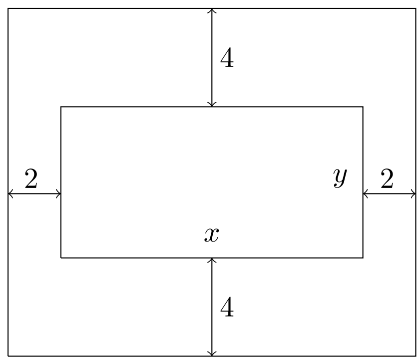
4. **(25 points)** A rectangular poster is to contain $50\,\text{in}^2$ of printed matter with margins of 4 inches at each of the top and bottom, and margins of 2 inches on each side. What are the height and width of the poster fitting those requirements that has the smallest possible area?

Solution. Here's the diagram:



The printed matter inside the poster is a rectangle; call this smaller rectangle's width x and its height y. Taking the margins into account, the full poster has width $x + 4$ and height $y + 8$.

So the printed area is $50 = xy$, which means $y = 50/x$.

Meanwhile, the full poster has area $(x + 4)(y + 8) = (x + 4)(50/x + 8) = 8x + 82 + 200/x$.
We have $x > 0$ and $y > 0$, which gives just $x > 0$. [Note that $x = 0$ is impossible to get $xy = 50$. And $y > 0$ gives only $50/x > 0$, which is the same as $x > 0$.]

So we must minimize $f(x) = 8x + 82 + 200x^{-1}$ on $(0, \infty)$.

We compute $f'(x) = 8 - 200x^{-2}2$, which is always defined on the original domain.

Setting $f'(x) = 0$ gives $8x^2 = 200$, so $x^2 = 25$, and so $x = \pm 5$; but $-5 \notin (0, \infty)$, meaning that the only critical point is $x = 5$. Our f' chart is:

x	$(0, 5)$	$(5, \infty)$
$f'(x)$	$-$	$+$
$f(x)$	$\searrow$	$\nearrow$

So by FDTAE, f has an absolute minimum at $x = 5\,\text{in}$. That gives $y = 50/5 = 10\,\text{in}$.

So the best poster therefore has width $x + 4 = 9\,\text{in}$, and height $y + 8 = 18\,\text{in}$.
That is, the best poster is $\boxed{9\,\text{in wide by } 18\,\text{in high}}$

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5. **(10 points)** Here are some values of a certain continuous function $h(x)$:

x	-4	-3	-2	-1	0	1	2	3	4	5	6
$h(x)$	3	1	0	-1	-2	-2	0	1	5	8	7

Estimate $\displaystyle\int_{-3}^{5} h(x)\,dx$ using **four** approximating rectangles of equal width and **right** endpoints. That is, compute R_4.

Solution. Cutting the interval $[-3,5]$ into four equal intervals means each interval has width $\Delta x = (5-(-3))/4 = 8/4 = 2$. Thus, the right endpoint Riemann sum for four intervals is
$R_4 = 2 \cdot [f(-1)+f(1)+f(3)+f(5)] = 2(-1-2+1+8) = 2 \cdot 6 = \boxed{12}$

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6. **(15 points)** Compute the following definite and indefinite integrals.

(a) $\displaystyle\int (5\sec t + 7\tan t)\sec t\, dt$ (b) $\displaystyle\int_{-1}^{2} x^3(x+3)^2\, dx$

Solution. (a) $\displaystyle\int (5\sec t + 7\tan t)\sec t\, dt = \int 5\sec^2 t + 7\sec t\tan t\, dt = \boxed{5\tan t + 7\sec t + C}$

(b) $\displaystyle\int_{-1}^{2} x^3(x+3)^2\, dx = \int_{-1}^{2} x^3(x^2+6x+9)\, dx = \int_{-1}^{2} x^5+6x^4+9x^3\, dx = \frac{1}{6}x^6+\frac{6}{5}x^5+\frac{9}{4}x^4\bigg|_{-1}^{2}$

$$= \left(\frac{1}{6}(64)+\frac{6}{5}(32)+\frac{9}{4}(16)\right) - \left(\frac{1}{6}-\frac{6}{5}+\frac{9}{4}\right) = \frac{63}{6}+\frac{6\cdot 33}{5}+\frac{9\cdot 15}{4}$$

[And you can just stop there. But for the record, that is:]

$$= \frac{21}{2}+\frac{198}{5}+\frac{135}{4} = \frac{210+792+675}{20} = \boxed{\frac{1677}{20}}$$