Solutions to Practice Problems for the Final Exam

1. Simplify each of the following expressions:

(a) $\ln(e^{\ln e})$ (b) $\ln\left|\ln\frac{1}{e}\right|$

Solutions. (a) $\ln(e^{\ln e}) = \ln e = \boxed{1}$

(b): $\ln\left|\ln\frac{1}{e}\right| = \ln|-1| = \ln 1 = \boxed{0}$

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2. Solve each of the the following equations for x:

(a) $\ln(\ln x) = 1$ (b) $\ln(x^2) = 2 + \ln x$ (c) $e^{3x-4} = 7$

Solutions. (a): $\ln(\ln x) = 1 \Leftrightarrow \ln x = e^1 = e \Leftrightarrow \boxed{x = e^e}$

(b): $\ln(x^2) = 2 + \ln x \Leftrightarrow 2\ln x = 2 + \ln x \Leftrightarrow \ln x = 2 \Leftrightarrow \boxed{x = e^2}$

(c): $e^{3x-4} = 7 \Leftrightarrow 3x - 4 = \ln 7 \Leftrightarrow 3x = 4 + \ln 7 \Leftrightarrow \boxed{x = \frac{4}{3} + \frac{1}{3}\ln 7}$

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3. Decide whether each statement is True or False. Explain why or why not.

(a) $(e^x)^2 = e^{x^2}$ (b) $\ln 5 - \ln 3 = \ln 2$ (c) $(\ln x)(\ln x) = \ln(x^2)$

Solutions. (a) $\boxed{\text{FALSE}}$

$(e^x)^2 = e^{2x}$ by exponent rules. This is different from e^{x^2}; for example with $x = 3$, we have $e^{2x} = e^6$, but $e^{x^2} = e^9 \neq e^6$

(b) $\boxed{\text{FALSE}}$

$\ln 5 - \ln 3 = \ln\left(\frac{5}{3}\right) \neq \ln 2$, since $\frac{5}{3} \neq 2$.

(c) $\boxed{\text{FALSE}}$

We have $\ln(x^2) = 2\ln x \neq (\ln x)(\ln x)$.

For example if $\ln x = 1$ (i.e., if $x = e$), then $2\ln x = 2(1) = 2$, but $(\ln x)(\ln x) = 1 \cdot 1 = 1$.

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4. Compute the derivatives of the following functions. (Hint: You may want to simplify first.)

(a) $f(x) = \ln(5xe^{-5x})$ (b) $g(x) = e^{\left(\ln(x^2+x) - \ln x\right)}$ (c) $h(x) = \ln\left(\frac{xe^x}{\sqrt{e^{7x}}}\right)$

Solutions. (a) $f(x) = \ln 5 + \ln x + \ln\left(e^{-5x}\right) = \ln 5 + \ln x - 5x$. Thus, $f'(x) = \boxed{\frac{1}{x} - 5}$

(b) $g(x) = e^{\left(\ln(x^2+x) - \ln x\right)} = \frac{e^{\ln(x^2+x)}}{e^{lnx}} = \frac{x^2+x}{x} = x + 1$. Thus, $g'(x) = \boxed{1}$

(c) $h(x) = \ln\left(\frac{xe^x}{\sqrt{e^{7x}}}\right) = \ln(xe^x) - \ln\sqrt{e^{7x}} = \ln x + \ln(e^x) - \frac{1}{2}\ln(e^{7x}) = \ln x + x - \frac{1}{2}(7x) = \ln x - \frac{5}{2}x$

Thus, $h'(x) = \boxed{\frac{1}{x} - \frac{5}{2}}$

5. Find the equation of the tangent line to the curve $y = (x+2)e^{-x}$ at the point $(0, 2)$.

Solution. We compute $y' = (x+2)e^{-x}(-1) + e^{-x}(1) = e^{-x}(1-(x+2)) = e^{-x}(1-x-2) = e^{-x}(-1-x)$

So the slope at $x = 0$ is given by $y'(0) = e^0(-1-0) = -1$.

Thus, the equation of the tangent line is $y - 2 = (-1)(x-0)$, or $\boxed{y = -x + 2}$

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6. Find the equation of the tangent line to the curve $y = \ln(xe^{-3x})$ at the point $(1, -3)$.

Solution. First, we can simplify the function $y = \ln(xe^{-3x}) = \ln x + \ln e^{-3x} = \ln x - 3x$.

So $y' = \dfrac{1}{x} - 3$.

So the slope at $x = 1$ is given by $y'(1) = 1 - 3 = -2$.

Thus, the equation of the tangent line is $y + 3 = -2(x-1)$, or $\boxed{y = -2x - 1}$

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7. Find an equation for the line tangent to $y = 4\sqrt{\ln x}$ at the point where $x = e$.

Solution. We have $y' = 4 \cdot \dfrac{1}{2}(\ln x)^{-1/2} \cdot \dfrac{1}{x} = \dfrac{2}{x\sqrt{\ln x}}$.

So the slope at $x = e$ is $y'(e) = \frac{2}{e\sqrt{1}} = \frac{2}{e}$.

Meanwhile, when $x = e$, we have $y(e) = 4\sqrt{\ln e} = 4\sqrt{1} = 4$. So the point is $(e, 4)$.

So by point-slope, the tangent line is $y - 4 = \dfrac{2}{e}(x-e)$, which simplifies to $y - 4 = \dfrac{2}{e}x - 2$,

i.e., $\boxed{y = \dfrac{2}{e}x + 2}$

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8. Let $y = \dfrac{\ln x}{1+x^2}$, find $f'(1)$.

Solution. By the Quotient Rule, $y' = \dfrac{(1+x^2)\left(\frac{1}{x}\right) - \ln x(2x)}{(1+x^2)^2} = \dfrac{\frac{1}{x} + x - 2x\ln x}{(1+x^2)^2}$.

So the slope at $x = 1$ is given by $y'(1) = \dfrac{1 + 1 - 2\ln 1}{4} = \boxed{\dfrac{1}{2}}$

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9. Let $f(x) = x\ln x$ with $x > 0$. Where is $f(x)$ concave up?

Solution. We compute $f'(x) = x\dfrac{1}{x} + \ln x(1) = 1 + \ln x$.

So $f''(x) = \dfrac{1}{x}$.

We have that $f''(x) > 0$ for $x > 0$. (Note also that the *formula* for f'' has $f''(x) < 0$ for $x < 0$, but the original domain of f is only $x > 0$.)

So $f''(x) > 0$ for all x in the domain $x > 0$ of f.

That is, $\boxed{f \text{ is concave up on its whole domain } (0, \infty)}$

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10. Let $x^2 e^y = \ln(xy)$. Find $\dfrac{dy}{dx}$.

Solution. Implicit differentiation says $\dfrac{d}{dx}\left(x^2 e^y\right) = \dfrac{d}{dx}\left(\ln(xy)\right)$.

That is, $x^2 e^y \dfrac{dy}{dx} + e^y 2x = \dfrac{1}{xy}\left(x\dfrac{dy}{dx} + y\right)$, so $x^3 y e^y \dfrac{dy}{dx} + 2x^2 y e^y = x\dfrac{dy}{dx} + y$.

Rearranging ing yields $x^3 y e^y \dfrac{dy}{dx} - x\dfrac{dy}{dx} = y - 2x^2 y e^y$. Thus, $\boxed{\dfrac{dy}{dx} = \dfrac{y - 2x^2 y e^y}{x^3 y e^y - x}}$

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11. Find all critical numbers of the function $f(x) = (x^2 - 7)e^{-x}$, and classify each as local maximum, local minimum, or neither.

Solution. We compute $f'(x) = (x^2 - 7)e^{-x}(-1) + e^{-x}(2x) = e^{-x}(-x^2 + 7 + 2x)$, which is **always defined**.

Solving $f'(x) = 0$ gives $-x^2 + 2x + 7 = 0$ which, by the quadratic formula, has solutions $x = 1 \pm 2\sqrt{2}$. So these are our only two critical numbers. Our f' chart is:

x	$(-\infty, 1 - 2\sqrt{2})$	$(1 - 2\sqrt{2}, 1 + 2\sqrt{2})$	$(1 + 2\sqrt{2}, \infty)$
$f'(x)$	$-$	$+$	$-$
$f(x)$	$\searrow$	$\nearrow$	$\searrow$

So f has a $\boxed{\text{local maximum at } 1 + 2\sqrt{x}}$ and a $\boxed{\text{local minimum at } 1 - 2\sqrt{x}}$

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12. Let $f(x) = x^4 e^{-x}$. For this function, discuss domain, vertical and horizontal asymptote(s), interval(s) of increase or decrease, local extreme value(s), concavity, and inflection point(s). Then use this information to present a detailed and labelled sketch of the curve.
Take my word that $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = +\infty$

Solution. **Domain**: all of $(-\infty, \infty)$, since x^4 and e^{-x} are defined everywhere.

V.A.: No vertical asymptotes. (No zeros in a denominator.)

H.A.: By the "take my word," since $\lim_{x\to\infty} f(x) = 0$, there is a horizontal asymptote at $y = 0$ on the right. (But not on the left.)

First Derivative Information:
We compute $f'(x) = x^4 e^{-x}(-1) + e^{-x}(4x^3) = e^{-x}(4x^3 - x^4) = x^3(4 - x)e^{-x}$, which is always defined

Solving $f' = 0$ gives $x = 0, 4$. Our f' chart is

x	$(-\infty, 0)$	$(0, 4)$	$(4, \infty)$
$f'(x)$	$-$	$+$	$-$
$f(x)$	$\searrow$	$\nearrow$	$\searrow$

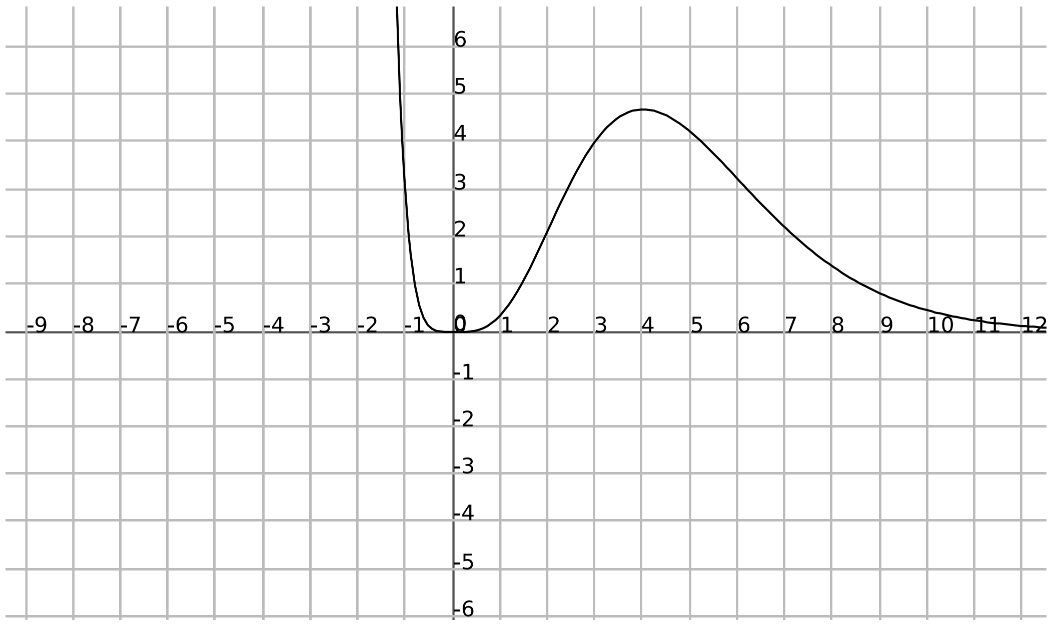
So f is increasing on the interval $(0, 4)$; and f is decreasing on $(-\infty, 0)$ and $(4, \infty)$. Moreover, f has a local max at $x = 4$ and a local min at $x = 0$.

Second Derivative Information:
Recall, $f'(x) = e^{-x}(4x^3 - x^4)$. So
$f''(x) = e^{-x}(12x^2 - 4x^3) - e^{-x}(4x^3 - x^4) = e^{-x}(12x^2 - 4x^3 - 4x^3 + x^4) = e^{-x}x^2(12 - 8x + x^2) = x^2(x - 2)(x - 6)e^{-x}$, which is always defined. Solving $f'' = 0$ gives $x = 0, 2, 6$. Our f'' chart is

x	$(-\infty, 0)$	$(0, 2)$	$(2, 6)$	$(6, \infty)$
$f'(x)$	$+$	$+$	$-$	$+$
$f(x)$	$\cup$	$\cup$	$\cap$	$\cup$

So f is concave down on the interval $(2, 6)$ and concave up on $(-\infty, 2)$ and $(6, \infty)$, with inflection points at $x = 2$ and $x = 6$.

Here is a graph:



Integrals: Evaluate the following definite and indefinite integrals.

13. $\int (x-3)\sqrt{x^2-6x+\pi}\,dx$

Solution. $\int (x-3)\sqrt{x^2-6x+\pi}\,dx \qquad [u = x^2-6x+\pi,\ du = (2x-6)\,dx,\ (x-3)\,dx = \frac{1}{2}\,du]$

$= \frac{1}{2}\int \sqrt{u}\,du = \frac{1}{2}\int u^{1/2}\,du = \frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C = \boxed{\frac{1}{3}(x^2-6x+\pi)^{3/2} + C}$

14. $\int \frac{y^3+y-1}{y^4+2y^2-4y+3}\,dy$

Solution. $\int \frac{y^3+y-1}{y^4+2y^2-4y+3}\,dy \quad [u = y^4+2y^2-4y+3,\ du = (4y^3+4y-4)\,dy,\ (y^3+y-1)\,dy = \frac{1}{4}\,du]$

$= \frac{1}{4}\int \frac{1}{u}\,du = \frac{1}{4}\ln|u| + C = \boxed{\frac{1}{4}\ln(y^4+2y^2-4y+3) + C}$

15. $\int_1^2 \frac{(x+1)(x-1)}{x^3}\,dx$

Solution. $\int_1^2 \frac{(x+1)(x-1)}{x^3}\,dx = \int_1^2 \frac{x^2-1}{x^3}\,dx = \int_1^2 x^{-1} - x^{-3}\,dx = \left[\ln x + \frac{1}{2}x^{-2}\right]_1^2$

$= \left[\ln 2 + \frac{1}{2}\cdot\frac{1}{4}\right] - \left[\ln 1 + \frac{1}{2}\cdot\frac{1}{1}\right] = \ln 2 + \frac{1}{8} - \frac{1}{2} = \boxed{\ln 2 - \frac{3}{8}}$

16. $\int \sec^2(3x)\sin(\tan(3x))\,dx$

Solution. $\int \sec^2(3x)\sin(\tan(3x))\,dx = \qquad [u = \tan(3x), du = 3\sec^2(3x)\,dx, \sec^2(3x)\,dx = \frac{1}{3}\,du]$

$= \frac{1}{3}\int \sin u\,du = -\frac{1}{3}\cos u + C = \boxed{-\frac{1}{3}\cos(\tan(3x)) + C}$

17. $\int_0^2 \frac{x}{\sqrt{2x^2+1}}\,dx$

Solution. $\displaystyle\int_0^2 \frac{x}{\sqrt{2x^2+1}}\,dx = \qquad [u = 2x^2+1, du = 4x\,dx, x\,dx = \frac{1}{4}\,du]$

$\displaystyle = \frac{1}{4}\int_1^9 u^{-1/2}\,du = \frac{1}{2}\left[u^{1/2}\right]_1^9 = \frac{1}{2}\left[\sqrt{9}-\sqrt{1}\right] = \frac{3-1}{2} = \boxed{1}$

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18. $\displaystyle\int (x-1)\csc^2(x^2-2x)\,dx$

Solution. $\displaystyle\int (x-1)\csc^2(x^2-2x)\,dx = \qquad [u = x^2-2x, du = (2x-2)\,dx, (x-1)\,dx = \frac{1}{2}\,du]$

$\displaystyle = \frac{1}{2}\int \csc^2 u\,du = -\frac{1}{2}\cot u + C = \boxed{-\frac{1}{2}\cot(x^2-2x)+C}$

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19. $\displaystyle\int_2^4 \frac{1}{x^2}\cos\left(\frac{\pi}{x}\right)\,dx$

Solution. $\displaystyle\int_2^4 \frac{1}{x^2}\cos\left(\frac{\pi}{x}\right)\,dx = \qquad [u = \frac{\pi}{x}, du = -\frac{\pi}{x^2}\,dx, \frac{1}{x^2}\,dx = -\frac{1}{\pi}\,du]$

$\displaystyle = -\frac{1}{\pi}\int_{\pi/2}^{\pi/4}\cos u\,du = -\frac{1}{\pi}\left[\sin u\right]_{\pi/2}^{\pi/4} = -\frac{1}{\pi}\left[\frac{\sqrt{2}}{2}-1\right] = \boxed{\frac{1}{\pi}-\frac{\sqrt{2}}{2\pi}}$

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20. $\displaystyle\int_0^{\pi/2} (\sin x + \cos x)^2\,dx$

Solution. $\displaystyle\int_0^{\pi/2} (\sin x + \cos x)^2\,dx = \int_0^{\pi/2} \sin^2 x + 2\sin x\cos x + \cos^2 x\,dx$

$\displaystyle = \int_0^{\pi/2} 1 + 2\sin x\cos x\,dx = \int_0^{\pi/2} 1\,dx + \int_0^{\pi/2} 2\sin x\cos x\,dx$

$[u = \sin x, du = \cos x\,dx$ on second integral only$]$ $\displaystyle\qquad = [x]_0^{\pi/2} + \int_0^1 2u\,du$

$\displaystyle = \left[\frac{\pi}{2}-0\right] + \left[u^2\right]_0^1 = \boxed{\frac{\pi}{2}+[1-0] = \frac{\pi}{2}+1}$

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21. $\displaystyle\int_1^e \frac{\sin(\pi\ln x)}{x}\,dx$

Solution. $\displaystyle\int_1^e \frac{\sin(\pi\ln x)}{x}\,dx = \qquad [u = \pi\ln x, du = \frac{\pi}{x}\,dx, \frac{dx}{x} = \frac{du}{\pi}]$

$\displaystyle = \frac{1}{\pi}\int_0^{\pi}\sin u\,du = \frac{1}{\pi}[\cos u]_0^{\pi} = \frac{1}{\pi}[1-(-1)] = \boxed{\frac{2}{\pi}}$

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22. $\displaystyle\int e^{2x}\cos(e^{2x}+1)\,dx$

Solution. $\displaystyle\int e^{2x}\cos(e^{2x}+1)\,dx = \qquad [u = e^{2x}+1, du = 2e^{2x}\,dx, e^{2x}\,dx = \frac{1}{2}\,du]$

$\displaystyle = \frac{1}{2}\int \cos u\,du = \frac{1}{2}\sin u + C = \boxed{\frac{1}{2}\sin(e^{2x}+1)+C}$

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23. $\displaystyle\int x(x^2+1)^{14}\,dx$

Solution. $\displaystyle\int x(x^2+1)^{14}\,dx \qquad [u = x^2+1,\ du = 2x\,dx,\ x\,dx = \frac{1}{2}\,du]$

$\displaystyle = \frac{1}{2}\int u^{14}\,du = \frac{1}{2}\frac{u^{15}}{15} + C = \boxed{\frac{1}{30}(x^2+1)^{15}+C}$

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24. $\int \sin(4x)\cos(4x)\ dx$

Solution. $\int \sin(4x)\cos(4x)\ dx \qquad [u=\sin(4x),\ du = 4\cos(4x)\,dx,\ \cos(4x)\,dx = \frac{1}{4}\,du]$

$= \frac{1}{4}\int u\ du = \frac{1}{4}\frac{u^2}{2} + C = \boxed{\frac{1}{8}\sin^2(4x) + C}$

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25. $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\ dx$

Solution. $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\ dx \qquad [u = e^x + e^{-x},\ du = (e^x - e^{-x})\,dx]$

$= \int \frac{1}{u}\ du = \ln|u| + C = \boxed{\ln|e^x + e^{-x}| + C}$

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26. $\int_1^{e^3} \frac{1}{x}\sqrt{1+\ln x}\ dx$

Solution. $\int_1^{e^3} \frac{1}{x}\sqrt{1+\ln x}\ dx \qquad [u = 1+\ln x,\ du = \frac{1}{x}\,dx;\ x = e^3 \Rightarrow u = 4;\ x = 1 \Rightarrow u = 1]$

$= \int_1^4 \sqrt{u}\ du = \frac{2}{3}u^{3/2}\Big|_1^4 = \frac{2}{3}\left(4^{3/2} - 1^{3/2}\right) = \frac{2}{3}(8-1) = \boxed{\frac{14}{3}}$

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27. $\int \frac{1}{(x+1)\ln(x+1)}\ dx$

Solution. $\int \frac{1}{(x+1)\ln(x+1)}\ dx \qquad [u = \ln(x+1),\ du = \frac{1}{x+1}\,dx]$

$= \int \frac{1}{u}\ du = \ln|u| + C = \boxed{\ln|\ln(x+1)| + C}$

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28. $\int \frac{\sin x}{7+\cos x}\ dx$

Solution. $\int \frac{\sin x}{7+\cos x}\ dx \qquad [u = 7+\cos x,\ du = -\sin x\,dx,\ \sin x\,dx = -du]$

$= -\int \frac{1}{u}\ du = -\ln|u| + C = \boxed{-\ln|7+\cos x| + C}$

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29. $\int \frac{6e^x}{e^x+7}\ dx$

Solution. $\int \frac{6e^x}{e^x+7}\ dx \qquad [u = e^x + 7,\ du = e^x\ dx]$

$= 6\int \frac{1}{u}\ dx = 6\ln|u| + C = \boxed{6\ln|e^x+7| + C}$

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30. $\int \frac{e^{\ln(\sin x)}}{e^{\ln(\cos x+7)}}\ dx$

Solution. $\int \frac{e^{\ln(\sin x)}}{e^{\ln(\cos x+7)}}\ dx = \int \frac{\sin x}{\cos x + 7}\ dx \qquad [u = 7+\cos x,\ du = -\sin x\,dx,\ \sin x\,dx = -du]$

$= -\int \frac{1}{u}\ du = -\ln|u| + C = \boxed{-\ln|7+\cos x| + C}$

[This is just number 28 again in disguise.]

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31. $\int \ln(e^{x^2}e^xe^7)\ dx$

Solution. $\int \ln(e^{x^2} e^x e^7)\ dx = \int x^2 + x + 7\, dx = \boxed{\frac{1}{3}x^3 + \frac{1}{2}x^2 + 7x + C}$

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32. $\int \frac{6x+3}{x^2+x-5}\ dx$

Solution. $\int \frac{6x+3}{x^2+x-5}\ dx [u = x^2 + x - 5,\ du = (2x+1)\, dx\, (6x+3)\, dx = 3\, du]$

$= \int \frac{3\, du}{u} = 3 \ln|u| + C = \boxed{3 \ln|x^2 + x - 5| + C}$

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33. $\int \frac{1}{1-2x}\ dx$

Solution. $\int \frac{1}{1-2x}\ dx \qquad [u = 1 - 2x,\ du = -2\, du,\ du = -\frac{1}{2}\, dx]$

$= -\frac{1}{2} \int \frac{du}{u} = -\frac{1}{2} \ln|u| + C = \boxed{-\frac{1}{2} \ln|1 - 2x| + C}$

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34. $\int e^{3x+1}\ dx$

Solution. $\int e^{3x+1}\ dx \qquad [u = 3x + 1,\ du = 3\, dx,\ dx = \frac{1}{3}\, du]$

$= \frac{1}{3} \int e^u\, du = \frac{1}{3} e^u + C = \boxed{\frac{1}{3} e^{3x+1} + C}$

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35. $\int \frac{e^{-1/x^7}}{x^8}\ dx$

Solution. $\int \frac{e^{-1/x^7}}{x^8}\ dx \qquad [u = -\frac{1}{x^7} = -x^{-7},\ du = 7x^{-8}\, dx,\ \frac{dx}{x^8} = \frac{1}{7}\, du]$

$= \frac{1}{7} \int e^u\ du = \frac{1}{7} e^u + C = \boxed{\frac{1}{7} e^{-1/x^7} + C}$

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36. $\int \frac{1}{e^x}\ dx$

Solution. $\int \frac{1}{e^x}\ dx = \int e^{-x}\ dx \qquad [u = -x,\ du = -dx,\ dx = -du]$

$= -\int e^u\ du = -e^u + C = \boxed{-e^{-x} + C}$

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37. $\int_0^1 \frac{1}{7x+1}\ dx$

Solution. $\int_0^1 \frac{1}{7x+1}\ dx \qquad [u = 7x + 1,\ du = 7\, dx,\ dx = \frac{1}{7}\, du;\ x = 0 \Rightarrow u = 1,\ x = 1 \Rightarrow u = 8]$

$= \int_1^8 \frac{1}{7} \frac{du}{u} = \frac{1}{7} \ln|u| \Big|_1^8 = \frac{1}{7}\big(\ln 8 - \ln 1\big) = \boxed{\frac{1}{7} \ln 8}$

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38. $\int_e^{e^2} \frac{1}{x(\ln x)^2}\ dx$

Solution. $\int_e^{e^2} \frac{1}{x(\ln x)^2}\ dx \qquad [u = \ln x,\ du = \frac{1}{x}\, dx;\ x = e \Rightarrow u = 1,\ x = e^2 \Rightarrow u = 2]$

$= \int_1^2 \frac{1}{u^2}\, du = -\frac{1}{u}\Big|_1^2 = -\frac{1}{2} - \big(-1\big) = \boxed{\frac{1}{2}}$

39. $\displaystyle\int_{\ln 4}^{\ln 7} 9e^{2x}\ dx$

Solution. $\displaystyle\int_{\ln 4}^{\ln 7} 9e^{2x}\ dx \quad [u = 2x,\ du = 2\,dx,\ dx = \frac{1}{2}\,du;\ x = \ln 4 \Rightarrow u = 2\ln 4,\ x = \ln 7 \Rightarrow u = 2\ln 7]$

$\displaystyle = \int_{2\ln 4}^{2\ln 7} \frac{9}{2}e^{u}\,du = \frac{9}{2}e^{u}\Big|_{2\ln 4}^{2\ln 7} = \frac{9}{2}\big(e^{2\ln 7} - e^{2\ln 4}\big) = \frac{9}{2}\Big(\big(e^{\ln 7}\big)^2 - \big(e^{\ln 4}\big)^2\Big)$

$\displaystyle = \frac{9}{2}\big(7^2 - 4^2\big) = \frac{9}{2}\cdot 33 = \boxed{\frac{297}{2}}$

40. $\displaystyle\int_{0}^{\ln 3} \left(2 + \frac{1}{e^x}\right)^2\ dx$

Solution. $\displaystyle\int_{0}^{\ln 3} \left(2 + \frac{1}{e^x}\right)^2\ dx = \int_{0}^{\ln 3} 4 + 4e^{-x} + e^{-2x}\,dx = 4x - 4e^{-x} - \frac{1}{2}e^{-2x}\Big|_{0}^{\ln 3}$

$\displaystyle = \left(4\ln 3 - 4e^{-\ln 3} - \frac{1}{2}e^{-2\ln 3}\right) - \left(0 - 4e^{0} - \frac{1}{2}e^{0}\right) = 4\ln 3 - \frac{4}{3} - \frac{1}{2}\cdot\frac{1}{3^2} + 4 + \frac{1}{2} = \boxed{4\ln 3 + \frac{28}{9}}$

41. $\displaystyle\int \frac{we^{w^2}}{17 + e^{w^2}}\ dw$

Solution. $\displaystyle\int \frac{we^{w^2}}{17 + e^{w^2}}\ dw \qquad [u = 17 + e^{w^2},\ du = 2we^{w^2}\ dw,\ we^{w^2}\ dw = \frac{1}{2}\,du]$

$\displaystyle = \frac{1}{2}\int \frac{1}{u}\ du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|17 + e^{w^2}| + C = \boxed{\frac{1}{2}\ln\left(17 + e^{w^2}\right) + C}$

42. $\displaystyle\int_{\ln 2}^{\ln 3} e^{2x}\ dx$

Solution. $\displaystyle\int_{\ln 2}^{\ln 3} e^{2x}\ dx = \frac{1}{2}e^{2x}\Big|_{\ln 2}^{\ln 3} = \frac{1}{2}\big(e^{2\ln 3} - e^{2\ln 2}\big) = \frac{1}{2}\big(3^2 - 2^2\big) = \boxed{\frac{5}{2}}$

43. $\displaystyle\int \frac{e^{-x}\ln(1 + e^{-x})}{1 + e^{-x}}\ dx$

Solution. $\displaystyle\int \frac{e^{-x}\ln(1 + e^{-x})}{1 + e^{-x}}\ dx \qquad [u = \ln(1+e^{-x}),\ du = \frac{1}{1 + e^{-x}}(e^{-x})(-1)\ dx,\ \frac{e^{-x}}{1 + e^{-x}}\ dx = -du]$

$\displaystyle = -\int u\ du = -\frac{u^2}{2} + C = \boxed{-\frac{(\ln(1 + e^{-x}))^2}{2} + C}$

44. $\displaystyle\int_{e}^{e^4} \frac{1}{x\sqrt{\ln x}}\ dx$

Solution. $\displaystyle\int_{e}^{e^4} \frac{1}{x\sqrt{\ln x}}\ dx \qquad [u = \ln x,\ du = \frac{1}{x}\,dx;\ x = e^4 \Rightarrow u = 4,\ x = e \Rightarrow u = 1]$

$\displaystyle = \int_{1}^{4} u^{-1/2}\,du 2u^{\frac{1}{2}}\Big|_{1}^{4} = 2\sqrt{4} - 2\sqrt{1} = 4 - 2 = \boxed{2}$

45. $\displaystyle\int (e^{3x} + e^{-7x})^2\ dx$

Solution. $\displaystyle\int (e^{3x} + e^{-7x})^2\,dx = \int e^{6x} + 2e^{-4x} + e^{-14x}\,dx = \boxed{\frac{1}{6}e^{6x} - \frac{1}{2}e^{-4x} - \frac{1}{14}e^{-14x} + C}$