

Solutions to Practice Problems 1

Functions: Please state what the domain is for each of the following functions.

1. $f(x) = \frac{x+2}{x-1}$

2. $g(x) = \sqrt{x-2}$

3. $m(x) = \sqrt{2-x}$

4. $G(x) = \frac{1}{\sqrt{2-x}}$

5. $h(x) = \frac{x-3}{x^2+3}$

6. $W(x) = \frac{x^2+6x+8}{x+2}$

Solutions to Domain problems:

1. Need $x \neq 1$, so Domain of f is all real numbers except 1
or if you prefer, $\{x \in \mathbb{R} | x \neq 1\}$, or $(-\infty, 1) \cup (1, \infty)$

2. Need $x - 2 \geq 0$, so Domain of g is $[2, \infty)$ or if you prefer, $\{x \in \mathbb{R} | x \geq 2\}$

3. Need $2 - x \geq 0$, so $x \leq 2$, so Domain of m is $(-\infty, 2]$ or if you prefer, $\{x \in \mathbb{R} | x \leq 2\}$

4. Need $2 - x \geq 0$, but also $2 - x \neq 0$, so $x < 2$, so Domain of G is $(-\infty, 2)$ or if you prefer, $\{x \in \mathbb{R} | x < 2\}$

5. Denom is never 0, so Domain of h is $(-\infty, \infty)$ or if you prefer, \mathbb{R}

6. Need $x \neq -2$, so Domain of f is all real numbers except -2
or if you prefer, $\{x \in \mathbb{R} | x \neq -2\}$, or $(-\infty, -2) \cup (-2, \infty)$

7. Let $g(x) = \frac{x+1}{x}$. Compute (and simplify, if possible) the following:

(a) $g(t-2) =$

(b) $\frac{g(2+h) - g(2)}{h} =$

Solution. (a) $g(t-2) = \frac{(t-2)+1}{t-2} = \frac{t-1}{t-2}$

(b) $\frac{g(2+h) - g(2)}{h} = \frac{\left(\frac{(2+h)+1}{2+h}\right) - \frac{3}{2}}{h} \cdot \frac{(2(2+h))}{(2(2+h))} = \frac{(3+h)2 - 3(2+h)}{2h(2+h)}$
 $= \frac{6+2h-6-3h}{2h(h+2)} = \frac{-h}{2h(h+2)} = \frac{-1}{2(h+2)}$

8. Let $f(x) = \frac{1}{x+1} - \frac{1}{x}$. Compute (and simplify, if possible) the following:

(a) $f(t-1) =$

(b) $f\left(\frac{1}{t}\right) =$

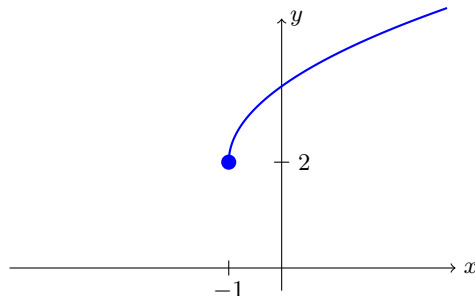
Solution. (a) $f(t-1) = \frac{1}{(t-1)+1} - \frac{1}{t-1} = \frac{1}{t} - \frac{1}{t-1} = \frac{(t-1)-t}{t(t-1)} = \frac{-1}{t(t-1)}$

(b) $f\left(\frac{1}{t}\right) = \frac{1}{\left(\frac{1}{t}+1\right)} - \frac{1}{\left(\frac{1}{t}\right)} = \frac{t}{t\left(\frac{1}{t}+1\right)} - t = \frac{t}{t+1} - t = \frac{t-t(t+1)}{t+1} = \frac{t-t^2-t}{t+1} = \frac{-t^2}{t+1}$

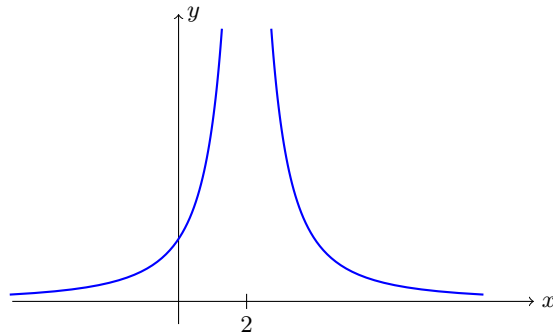
9. Graph the following functions using scaling, translation, etc.

(a) $y = 2 + \sqrt{x+1}$ (b) $y = \frac{3}{(x-2)^2}$ (c) $y = 2(x-1)^4 - 3$

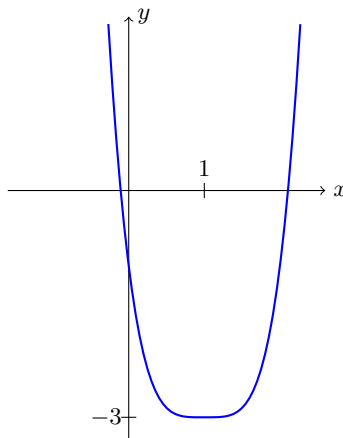
Solutions (a): This is $y = \sqrt{x}$ translated **left by 1** and then **up by 2**, so it looks like:



(b): This is $y = \frac{1}{x^2}$ translated **right by 2**, and then **stretched vertically by 3**, so it looks like:



(c): This is $y = x^4$ **stretched vertically by 2**, then translated **right by 1**, and then **down by 3**, so it looks like:



Limit Practice Problems: Evaluate the following limits. Always justify your work.

Solutions. 10. $\lim_{w \rightarrow 0} \frac{16}{w} = \frac{16}{0}$ so we check both sides:

$$\text{LHL: } \lim_{w \rightarrow 0^-} \frac{16}{w} = \frac{16}{0^-} = -\infty$$

$$\text{RHL: } \lim_{w \rightarrow 0^+} \frac{16}{w} = \frac{16}{0^+} = +\infty$$

RHL \neq LHL, so original limit DNE

11. $\lim_{t \rightarrow 2} \frac{3-t}{t-2} = \frac{1}{0}$ so we check both sides:

LHL: $\lim_{t \rightarrow 2^-} \frac{3-t}{t-2} = \frac{1}{0^-} = -\infty$

RHL: $\lim_{t \rightarrow 2^+} \frac{3-t}{t-2} = \frac{1}{0^+} = +\infty$

RHL \neq LHL, so original limit DNE

12. $\lim_{t \rightarrow 2} \frac{3-t}{(t-2)^2} = \frac{1}{0}$ so we check both sides:

LHL: $\lim_{t \rightarrow 2^-} \frac{3-t}{(t-2)^2} = \frac{1}{0^+} = +\infty$

RHL: $\lim_{t \rightarrow 2^+} \frac{3-t}{(t-2)^2} = \frac{1}{0^+} = +\infty$

RHL=LHL= ∞ , so original limit diverges to $+\infty$

13. $\lim_{x \rightarrow 4} \frac{(x+2)^2}{x^2-3x-4} = \frac{6^2}{0}$ so we check both sides, noting that the denominator factors as $(x-4)(x+1)$

LHL: $\lim_{x \rightarrow 4^-} \frac{(x+2)^2}{x^2-3x-4} = \frac{6^2}{(0^-) \cdot 5} = -\infty$

RHL: $\lim_{x \rightarrow 4^+} \frac{(x+2)^2}{x^2-3x-4} = \frac{6^2}{(0^+) \cdot 5} = +\infty$

RHL \neq LHL, so original limit DNE

14. $\lim_{x \rightarrow 4} \frac{x-4}{x^2-3x-4} = \lim_{x \rightarrow 4} \frac{x-4}{(x-4)(x+1)} = \lim_{x \rightarrow 4} \frac{1}{x+1} \stackrel{\text{DSP}}{=} \frac{1}{5}$

15. $\lim_{x \rightarrow 4} \frac{x^2-2x-8}{x^2-3x-4} = \lim_{x \rightarrow 4} \frac{(x-4)(x+2)}{(x-4)(x+1)} = \lim_{x \rightarrow 4} \frac{x+2}{x+1} \stackrel{\text{DSP}}{=} \frac{6}{5}$

16. $\lim_{x \rightarrow 6} \frac{x^2-4x-12}{x^2-3x-18} = \lim_{x \rightarrow 6} \frac{(x-6)(x+2)}{(x-6)(x+3)} = \lim_{x \rightarrow 6} \frac{x+2}{x+3} \stackrel{\text{DSP}}{=} \frac{8}{9}$

17. $\lim_{x \rightarrow 1} \frac{x^2-4x-12}{x^2-3x-18} \stackrel{\text{DSP}}{=} \frac{1-4-12}{1-3-18} = \frac{-15}{-20} = \frac{3}{4}$

18. $\lim_{x \rightarrow 0} \frac{x^2-4x-12}{x^2-3x-18} \stackrel{\text{DSP}}{=} \frac{-12}{-18} = \frac{2}{3}$

19. $\lim_{x \rightarrow -3} \frac{x^2-4x-12}{x^2-3x-18} = \frac{9}{0}$ so we check both sides, noting that as we saw in #16 the function reduces to $\frac{x+2}{x+3}$:

LHL: $\lim_{x \rightarrow -3^-} \frac{x+2}{x+3} = \frac{-1}{0^-} = +\infty$

RHL: $\lim_{x \rightarrow -3^+} \frac{x+2}{x+3} = \frac{-1}{0^+} = -\infty$

RHL \neq LHL, so original limit DNE

20. $\lim_{x \rightarrow -2} \frac{x^2-4x-12}{x^2-3x-18} \stackrel{\text{DSP}}{=} \frac{4+8-12}{4+6-18} = \frac{0}{-8} = 0$

21. $\lim_{x \rightarrow 0} \frac{x^2-4x-12}{x^2-7x} = \frac{-12}{0}$, so we check both sides, noting the the denominator factors as $x(x-7)$:

LHL: $\lim_{x \rightarrow 0^-} \frac{x^2-4x-12}{x^2-7x} = \frac{-12}{(0^-)(-7)} = -\infty$

RHL: $\lim_{x \rightarrow 0^+} \frac{x^2-4x-12}{x^2-7x} = \frac{-12}{(0^+)(-7)} = +\infty$

RHL \neq LHL, so original limit DNE

22. $\lim_{x \rightarrow 0} \frac{x^2-4x}{x^2-7x} = \lim_{x \rightarrow 0} \frac{x-4}{x-7} \stackrel{\text{DSP}}{=} \frac{-4}{-7} = \frac{4}{7}$

23. $\lim_{x \rightarrow 3} \frac{x^2 - 9}{|x - 3|}$ is piecewise, so check both sides:

$$\text{LHL: } \lim_{x \rightarrow 3^-} \frac{x^2 - 9}{|x - 3|} = \lim_{x \rightarrow 3^-} \frac{(x - 3)(x + 3)}{-(x - 3)} = \lim_{x \rightarrow 3^-} \frac{x + 3}{-1} \stackrel{\text{DSP}}{=} -6$$

$$\text{RHL: } \lim_{x \rightarrow 3^+} \frac{x^2 - 9}{|x - 3|} = \lim_{x \rightarrow 3^+} \frac{(x - 3)(x + 3)}{x - 3} = \lim_{x \rightarrow 3^+} \frac{x + 3}{1} \stackrel{\text{DSP}}{=} 6$$

RHL \neq LHL, so original limit DNE

24. $\lim_{x \rightarrow 0} \frac{x^3 + 209x^2 + 200x}{|x|}$ is piecewise, so check both sides:

$$\text{LHL: } \lim_{x \rightarrow 0^-} \frac{x^3 + 209x^2 + 200x}{|x|} = \lim_{x \rightarrow 0^-} \frac{x^3 + 209x^2 + 200x}{-x} = \lim_{x \rightarrow 0^-} \frac{x^2 + 209x + 200}{-1} \stackrel{\text{DSP}}{=} -200$$

$$\text{RHL: } \lim_{x \rightarrow 0^+} \frac{x^3 + 209x^2 + 200x}{x} = \lim_{x \rightarrow 0^+} \frac{x^3 + 209x^2 + 200x}{x} = \lim_{x \rightarrow 0^+} \frac{x^2 + 209x + 200}{1} \stackrel{\text{DSP}}{=} 200$$

RHL \neq LHL, so original limit DNE

25. $\lim_{x \rightarrow -5} \frac{x^2 + 6x + 5}{|x + 5|}$ is piecewise, so check both sides:

$$\text{LHL: } \lim_{x \rightarrow -5^-} \frac{x^2 + 6x + 5}{|x + 5|} = \lim_{x \rightarrow -5^-} \frac{(x + 1)(x + 5)}{-(x + 5)} = \lim_{x \rightarrow -5^-} -(x + 1) \stackrel{\text{DSP}}{=} 4$$

$$\text{RHL: } \lim_{x \rightarrow -5^+} \frac{x^2 + 6x + 5}{|x + 5|} = \lim_{x \rightarrow -5^+} \frac{(x + 1)(x + 5)}{x + 5} = \lim_{x \rightarrow -5^+} (x + 1) \stackrel{\text{DSP}}{=} -4$$

RHL \neq LHL, so original limit DNE

$$26. \lim_{t \rightarrow -1} \frac{200(t^2 + 6t + 5)}{t^2 + t} = \lim_{t \rightarrow -1} \frac{200(t + 1)(t + 5)}{t(t + 1)} = \lim_{t \rightarrow -1} \frac{200(t + 5)}{t} \stackrel{\text{DSP}}{=} \frac{200(4)}{-1} = \boxed{-800}$$

$$27. \lim_{t \rightarrow 1} t^{300} + t^{200} + t^{100} \stackrel{\text{DSP}}{=} 1 + 1 + 1 = \boxed{3}$$

$$28. \lim_{x \rightarrow 1} \frac{x - 1}{\sqrt{x + 3} - 2} = \lim_{x \rightarrow 1} \frac{(x - 1)(\sqrt{x + 3} + 2)}{(\sqrt{x + 3} - 2)(\sqrt{x + 3} + 2)} = \lim_{x \rightarrow 1} \frac{(x - 1)(\sqrt{x + 3} + 2)}{(x + 3) - 4}$$

$$= \lim_{x \rightarrow 1} \frac{(x - 1)(\sqrt{x + 3} + 2)}{x - 1} = \lim_{x \rightarrow 1} \sqrt{x + 3} + 2 \stackrel{\text{DSP}}{=} \sqrt{4} + 2 = \boxed{4}$$

$$29. \lim_{x \rightarrow 9} \frac{9x - x^2}{3 - \sqrt{x}} = \lim_{x \rightarrow 9} \frac{(9x - x^2)(3 + \sqrt{x})}{(3 - \sqrt{x})(3 + \sqrt{x})} = \lim_{x \rightarrow 9} \frac{x(9 - x)(3 + \sqrt{x})}{9 - x}$$

$$= \lim_{x \rightarrow 9} x(3 + \sqrt{x}) \stackrel{\text{DSP}}{=} 9(3 + \sqrt{9}) = 9(6) = \boxed{54}$$

$$30. \lim_{x \rightarrow 1} \frac{\sqrt{x^2 + 8} - 3}{x - 1} = \lim_{x \rightarrow 1} \frac{(\sqrt{x^2 + 8} - 3)(\sqrt{x^2 + 8} + 3)}{(x - 1)(\sqrt{x^2 + 8} + 3)} = \lim_{x \rightarrow 1} \frac{(x^2 + 8) - 9}{(x - 1)(\sqrt{x^2 + 8} + 3)}$$

$$= \lim_{x \rightarrow 1} \frac{x^2 - 1}{(x - 1)(\sqrt{x^2 + 8} + 3)} = \lim_{x \rightarrow 1} \frac{(x - 1)(x + 1)}{(x - 1)(\sqrt{x^2 + 8} + 3)} = \lim_{x \rightarrow 1} \frac{x + 1}{\sqrt{x^2 + 8} + 3}$$

$$\stackrel{\text{DSP}}{=} \frac{2}{\sqrt{9} + 3} = \frac{2}{6} = \boxed{\frac{1}{3}}$$

$$31. \lim_{x \rightarrow -4} \frac{x^2 - 3x - 28}{x^2 + 4x} = \lim_{x \rightarrow -4} \frac{(x + 4)(x - 7)}{x(x + 4)} = \lim_{x \rightarrow -4} \frac{x - 7}{x} \stackrel{\text{DSP}}{=} \frac{-11}{-4} = \boxed{\frac{11}{4}}$$

32. $\lim_{x \rightarrow 0} \frac{x^2 - 3x - 28}{x^2 + 4x} = \frac{-28}{0}$ so we check both sides, noting that as we saw in #31 the function reduces to $\frac{x-7}{x}$:

$$\text{LHL: } \lim_{x \rightarrow 0^-} \frac{x-7}{x} = \frac{-7}{0^-} = +\infty \qquad \text{RHL: } \lim_{x \rightarrow 0^+} \frac{x-7}{x} = \frac{-7}{0^+} = -\infty$$

RHL \neq LHL = ∞ , so original limit DNE

$$\begin{aligned} 33. \lim_{x \rightarrow 3} \frac{\frac{2}{x+3} - \frac{1}{3}}{x-3} &= \lim_{x \rightarrow 3} \frac{\frac{2}{x+3} - \frac{1}{3}}{x-3} \left(\frac{3(x+3)}{3(x+3)} \right) = \lim_{x \rightarrow 3} \frac{6 - (x+3)}{3(x-3)(x+3)} \\ &= \lim_{x \rightarrow 3} \frac{-(x-3)}{3(x-3)(x+3)} = \lim_{x \rightarrow 3} \frac{-1}{3(x+3)} \stackrel{\text{DSP}}{=} \frac{-1}{3(6)} = \boxed{-\frac{1}{18}} \end{aligned}$$

34. $\lim_{x \rightarrow 1} \frac{x^2 - 1}{|x - 1|}$ is piecewise, so check both sides:

$$\text{LHL: } \lim_{x \rightarrow 1^-} \frac{x^2 - 1}{|x - 1|} = \lim_{x \rightarrow 1^-} \frac{(x-1)(x+1)}{-(x-1)} = \lim_{x \rightarrow 1^-} \frac{x+1}{-1} \stackrel{\text{DSP}}{=} -2$$

$$\text{RHL: } \lim_{x \rightarrow 1^+} \frac{x^2 - 1}{|x - 1|} = \lim_{x \rightarrow 1^+} \frac{(x-1)(x+1)}{x-1} = \lim_{x \rightarrow 1^+} \frac{x+1}{1} \stackrel{\text{DSP}}{=} 2$$

RHL \neq LHL, so original limit DNE

35. $\lim_{x \rightarrow -5} \frac{x^2 + 6x + 5}{|x + 5|}$ is piecewise, so check both sides:

$$\text{LHL: } \lim_{x \rightarrow -5^-} \frac{x^2 + 6x + 5}{|x + 5|} = \lim_{x \rightarrow -5^-} \frac{(x+1)(x+5)}{-(x+5)} = \lim_{x \rightarrow -5^-} -(x+1) \stackrel{\text{DSP}}{=} 4$$

$$\text{RHL: } \lim_{x \rightarrow -5^+} \frac{x^2 + 6x + 5}{|x + 5|} = \lim_{x \rightarrow -5^+} \frac{(x+1)(x+5)}{x+5} = \lim_{x \rightarrow -5^+} (x+1) \stackrel{\text{DSP}}{=} -4$$

RHL \neq LHL, so original limit DNE

(And yes, this is the same problem as #25.)

36. $\lim_{x \rightarrow -1} \frac{x^2 + 3x + 2}{(x+1)^2} = \lim_{x \rightarrow -1} \frac{(x+1)(x+2)}{(x+1)^2} = \lim_{x \rightarrow -1} \frac{x+2}{x+1} = \frac{1}{0}$, so we check both sides:

$$\text{LHL: } \lim_{x \rightarrow -1^-} \frac{x+2}{x+1} = \frac{1}{0^-} = -\infty \qquad \text{RHL: } \lim_{x \rightarrow -1^+} \frac{x+2}{x+1} = \frac{1}{0^+} = +\infty$$

RHL \neq LHL, so original limit DNE

$$37. \lim_{x \rightarrow 7^-} \frac{7-x}{|x-7|} = \lim_{x \rightarrow 7^-} \frac{-(x-7)}{-(x-7)} = \lim_{x \rightarrow 7^-} 1 \stackrel{\text{DSP}}{=} \boxed{1}$$

$$38. \lim_{x \rightarrow 0^-} \frac{x}{x-|x|} = \lim_{x \rightarrow 0^-} \frac{x}{x-(-x)} = \lim_{x \rightarrow 0^-} \frac{x}{2x} = \lim_{x \rightarrow 0^-} \frac{1}{2} \stackrel{\text{DSP}}{=} \boxed{\frac{1}{2}}$$

$$39. \lim_{x \rightarrow 2^+} \frac{2-x}{|x-2|} = \lim_{x \rightarrow 2^+} \frac{-(x-2)}{x-2} = \lim_{x \rightarrow 2^+} -1 \stackrel{\text{DSP}}{=} \boxed{-1}$$

40. Let $G(u) = u^2 + u$. Compute $\lim_{u \rightarrow 2} \frac{u^2 - 2u}{G(u-3)}$

$$\begin{aligned} \text{Solution. } \lim_{u \rightarrow 2} \frac{u^2 - 2u}{G(u-3)} &= \lim_{u \rightarrow 2} \frac{u^2 - 2u}{(u-3)^2 + (u-3)} = \lim_{u \rightarrow 2} \frac{u(u-2)}{u^2 - 5u + 6} = \lim_{u \rightarrow 2} \frac{u(u-2)}{(u-3)(u-2)} \\ &= \lim_{u \rightarrow 2} \frac{u}{u-3} = \frac{2}{-1} = \boxed{-2} \end{aligned}$$

41. Let $h(y) = y^2 - 3$. Compute $\lim_{x \rightarrow -2} \frac{x+2}{h(2x) - h(x+6)}$

Solution.

$$\begin{aligned} \lim_{x \rightarrow -2} \frac{x+2}{h(2x) - h(x+6)} &= \lim_{x \rightarrow -2} \frac{x+2}{((2x)^2 - 3) - ((x+6)^2 - 3)} = \lim_{x \rightarrow -2} \frac{x+2}{(4x^2 - 3) - (x^2 + 12x + 36 - 3)} \\ &= \lim_{x \rightarrow -2} \frac{x+2}{4x^2 - 3 - x^2 - 12x - 33} = \lim_{x \rightarrow -2} \frac{x+2}{3x^2 - 12x - 36} = \lim_{x \rightarrow -2} \frac{x+2}{3(x^2 - 4x - 12)} \\ &= \lim_{x \rightarrow -2} \frac{x+2}{3(x-6)(x+2)} = \lim_{x \rightarrow -2} \frac{1}{3(x-6)} = \boxed{-\frac{1}{24}} \end{aligned}$$

42. Let $g(x) = \sqrt{x}$. Compute $\lim_{s \rightarrow 1} \frac{g(s^2 + 8) - 3}{s - 1}$

$$\begin{aligned} \text{Solution. } \lim_{s \rightarrow 1} \frac{g(s^2 + 8) - 3}{s - 1} &= \lim_{s \rightarrow 1} \frac{\sqrt{s^2 + 8} - 3}{s - 1} = \lim_{s \rightarrow 1} \frac{(\sqrt{s^2 + 8} - 3) \cdot (\sqrt{s^2 + 8} + 3)}{(s - 1) \cdot (\sqrt{s^2 + 8} + 3)} \\ &= \lim_{s \rightarrow 1} \frac{s^2 + 8 - 9}{(s - 1)(\sqrt{s^2 + 8} + 3)} = \lim_{s \rightarrow 1} \frac{s^2 - 1}{(s - 1)(\sqrt{s^2 + 8} + 3)} = \lim_{s \rightarrow 1} \frac{(s - 1)(s + 1)}{(s - 1)(\sqrt{s^2 + 8} + 3)} \\ &= \lim_{s \rightarrow 1} \frac{s + 1}{\sqrt{s^2 + 8} + 3} = \frac{2}{6} = \boxed{\frac{1}{3}} \end{aligned}$$

43. Let $f(t) = \frac{1}{t}$. Compute $\lim_{t \rightarrow 2} \frac{f(t-1) - 2f(t)}{t^2 - 4}$

$$\begin{aligned} \text{Solution. } \lim_{t \rightarrow 2} \frac{f(t-1) - 2f(t)}{t^2 - 4} &= \lim_{t \rightarrow 2} \frac{\left(\frac{1}{t-1} - \frac{2}{t}\right)}{t^2 - 4} = \lim_{t \rightarrow 2} \frac{\left(\frac{1}{t-1} - \frac{2}{t}\right) \cdot \left(\frac{t(t-1)}{t(t-1)}\right)}{t^2 - 4} \\ &= \lim_{t \rightarrow 2} \frac{t - 2(t-1)}{t(t-1)(t^2 - 4)} = \lim_{t \rightarrow 2} \frac{-(t-2)}{t(t-1)(t-2)(t+2)} = \lim_{t \rightarrow 2} \frac{-1}{t(t-1)(t+2)} \stackrel{\text{DSP}}{=} \frac{-1}{1 \cdot 2 \cdot 4} = \boxed{-\frac{1}{8}} \end{aligned}$$

Derivatives: Use the **limit definition of the derivative** to compute these derivatives:

44. $f(x) = -4x - x^2 - 3$ Find $f'(x)$

45. $g(x) = \frac{-3}{x}$ Find $g'(x)$

46. $R(x) = x^3$ Find $R'(x)$

47. $G(x) = \frac{1}{x^2}$ Find $G'(x)$

48. $f(x) = \sqrt{x-7}$ Find $f'(x)$

49. $g(x) = \sqrt{7-3x}$ Find $g'(x)$

Solutions to Derivative problems:

$$\begin{aligned} 44. f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{(-4(x+h) - (x+h)^2 - 3) - (-4x - x^2 - 3)}{h} = \\ &= \lim_{h \rightarrow 0} \frac{-4x - 4h - x^2 - 2xh - h^2 - 3 + 4x + x^2 + 3}{h} = \lim_{h \rightarrow 0} \frac{-4h - 2xh - h^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(-4 - 2x - h)}{h} = \lim_{h \rightarrow 0} -4 - 2x - h \stackrel{\text{DSP}}{=} \boxed{-4 - 2x} \end{aligned}$$

$$\begin{aligned} 45. g'(x) &= \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \rightarrow 0} \frac{\frac{-3}{x+h} - \left(\frac{-3}{x}\right)}{h} \cdot \left(\frac{x(x+h)}{x(x+h)}\right) = \lim_{h \rightarrow 0} \frac{-3x + 3(x+h)}{hx(x+h)} \\ &= \lim_{h \rightarrow 0} \frac{3h}{hx(x+h)} = \lim_{h \rightarrow 0} \frac{3}{x(x+h)} \stackrel{\text{DSP}}{=} \boxed{\frac{3}{x^2}} \end{aligned}$$

$$46. R'(x) = \lim_{h \rightarrow 0} \frac{R(x+h) - R(x)}{h} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{(x+h)^3 - x^3}{h}$$

$$\begin{aligned}
&= \lim_{h \rightarrow 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} = \lim_{h \rightarrow 0} \frac{3x^2h + 3xh^2 + h^3}{h} = \lim_{h \rightarrow 0} \frac{h(3x^2 + 3xh + h^2)}{h} \\
&= \lim_{h \rightarrow 0} 3x^2 + 3xh + h^2 \stackrel{\text{DSP}}{=} \boxed{3x^2}
\end{aligned}$$

$$\begin{aligned}
47. \quad G'(x) &= \lim_{h \rightarrow 0} \frac{G(x+h) - G(x)}{h} = \lim_{h \rightarrow 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h} = \lim_{h \rightarrow 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h} \cdot \left(\frac{x^2(x+h)^2}{x^2(x+h)^2} \right) \\
&= \lim_{h \rightarrow 0} \frac{x^2 - (x+h)^2}{h(x+h)^2x^2} = \lim_{h \rightarrow 0} \frac{x^2 - (x^2 + 2hx + h^2)}{h(x+h)^2x^2} = \lim_{h \rightarrow 0} \frac{-2hx - h^2}{h(x+h)^2x^2} \\
&= \lim_{h \rightarrow 0} \frac{-2x - h}{(x+h)^2x^2} \stackrel{\text{DSP}}{=} \frac{-2x}{(x^2)x^2} = \boxed{-\frac{2}{x^3}}
\end{aligned}$$

$$\begin{aligned}
48. \quad f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{(x+h)-7} - \sqrt{x-7}}{h} \\
&= \lim_{h \rightarrow 0} \frac{\sqrt{(x+h)-7} - \sqrt{x-7}}{h} \cdot \left(\frac{\sqrt{(x+h)-7} + \sqrt{x-7}}{\sqrt{(x+h)-7} + \sqrt{x-7}} \right) = \lim_{h \rightarrow 0} \frac{(x+h-7) - (x-7)}{h(\sqrt{(x+h)-7} + \sqrt{x-7})} \\
&= \lim_{h \rightarrow 0} \frac{x+h-7-x+7}{h(\sqrt{(x+h)-7} + \sqrt{x-7})} = \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{(x+h)-7} + \sqrt{x-7})} \\
&= \lim_{h \rightarrow 0} \frac{1}{\sqrt{(x+h)-7} + \sqrt{x-7}} \stackrel{\text{DSP}}{=} \boxed{\frac{1}{2\sqrt{x-7}}}
\end{aligned}$$

$$\begin{aligned}
49. \quad g'(x) &= \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{7-3(x+h)} - \sqrt{7-3x}}{h} \\
&= \lim_{h \rightarrow 0} \frac{\sqrt{7-3(x+h)} - \sqrt{7-3x}}{h} \cdot \left(\frac{\sqrt{7-3(x+h)} + \sqrt{7-3x}}{\sqrt{7-3(x+h)} + \sqrt{7-3x}} \right) = \lim_{h \rightarrow 0} \frac{(7-3(x+h)) - (7-3x)}{h(\sqrt{7-3(x+h)} + \sqrt{7-3x})} \\
&= \lim_{h \rightarrow 0} \frac{7-3x-3h-7+3x}{h(\sqrt{7-3(x+h)} + \sqrt{7-3x})} = \lim_{h \rightarrow 0} \frac{-3h}{h(\sqrt{7-3(x+h)} + \sqrt{7-3x})} \\
&= \lim_{h \rightarrow 0} \frac{-3}{\sqrt{7-3(x+h)} + \sqrt{7-3x}} \stackrel{\text{DSP}}{=} \boxed{\frac{-3}{2\sqrt{7-3x}}}
\end{aligned}$$

50. Find an equation for the tangent line to the graph of $f(x) = \frac{1}{x-1}$ at the point $(0, -1)$.

Solution. First, we find the slope $f'(0)$:

$$\begin{aligned}
f'(0) &= \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{\frac{1}{h-1} - \left(\frac{1}{-1}\right)}{h} \cdot \left(\frac{h-1}{h-1}\right) = \lim_{h \rightarrow 0} \frac{1 + (h-1)}{h(h-1)} \\
&= \lim_{h \rightarrow 0} \frac{h}{h(h-1)} = \lim_{h \rightarrow 0} \frac{1}{h-1} \stackrel{\text{DSP}}{=} -1
\end{aligned}$$

So the tangent line goes through $(0, -1)$ with slope -1 . By point-slope, the equation is:

$$y - (-1) = (-1)(x - 0), \quad \text{i.e.} \quad \boxed{y = -x - 1}$$

51. Find an equation for the tangent line to the graph of $g(x) = \frac{1}{x+1}$ at the point $\left(1, \frac{1}{2}\right)$.

Solution. First, we find the slope $g'(1)$:

$$\begin{aligned}
g'(1) &= \lim_{h \rightarrow 0} \frac{g(1+h) - g(1)}{h} = \lim_{h \rightarrow 0} \frac{\frac{1}{h+2} - \left(\frac{1}{2}\right)}{h} \cdot \left(\frac{2(h+2)}{2(h+2)}\right) = \lim_{h \rightarrow 0} \frac{2 - (h+2)}{2h(h+2)} \\
&= \lim_{h \rightarrow 0} \frac{-h}{2h(h+2)} = \lim_{h \rightarrow 0} \frac{-1}{2(h+2)} \stackrel{\text{DSP}}{=} -\frac{1}{4}
\end{aligned}$$

So the tangent line goes through $\left(1, \frac{1}{2}\right)$ with slope $-\frac{1}{4}$. By point-slope, the equation is:

$$y - \frac{1}{2} = -\frac{1}{4}(x - 1), \quad \text{i.e.} \quad \boxed{y = -\frac{x}{4} + \frac{3}{4}}$$

52. Find an equation for the tangent line to the graph of $y = \frac{3}{x} + 1$ when $x = 1$.

Solution. Let $f(x) = \frac{3}{x} + 1$. First, we find the slope $f'(1)$:

$$\begin{aligned} f'(1) &= \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \rightarrow 0} \frac{\left(\frac{3}{1+h} + 1\right) - \left(\frac{3}{1} + 1\right)}{h} \cdot \left(\frac{h+1}{h+1}\right) = \lim_{h \rightarrow 0} \frac{3 - 3(h+1)}{h(h+1)} \\ &= \lim_{h \rightarrow 0} \frac{-3h}{h(h+1)} = \lim_{h \rightarrow 0} \frac{-3}{h+1} \stackrel{\text{DSP}}{=} -3 \end{aligned}$$

We also have $f(1) = \frac{3}{1} + 1 = 4$. So the tangent line goes through $(1, 4)$ with slope -3 . By point-slope, the equation is:

$$y - 4 = -3(x - 1) \quad \text{i.e.} \quad \boxed{y = -3x + 7}$$

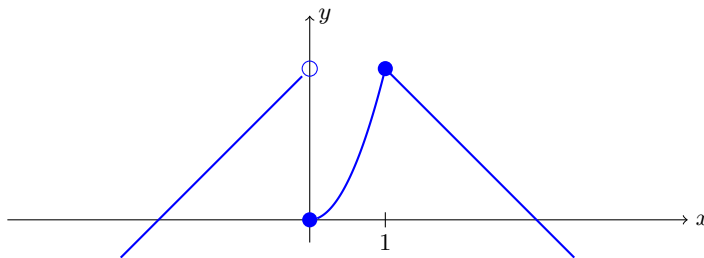
Piecewise-defined functions Answer the questions (and **justify** your answers) about each of the following piecewise defined functions.

$$53. \text{ Let } f(x) = \begin{cases} x + 2 & \text{if } x < 0 \\ 2x^2 & \text{if } 0 \leq x \leq 1 \\ 3 - x & \text{if } x > 1 \end{cases}$$

Sketch the graph. Find the numbers at which f is discontinuous. Evaluate:

$$\lim_{x \rightarrow 2} f(x) = \qquad \lim_{x \rightarrow 1} f(x) = \qquad \lim_{x \rightarrow 0} f(x) =$$

Solution. Using translation/scaling for the various pieces, and then putting them together, here's the graph:



For the limits:

$$\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} 3 - x = \boxed{1}$$

And since that's $f(2)$ also, that means f is continuous at $x = 2$

$\lim_{x \rightarrow 1} f(x)$: we check both sides:

$$\text{LHL: } \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} 2x^2 \stackrel{\text{DSP}}{=} 2$$

$$\text{RHL: } \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} 3 - x \stackrel{\text{DSP}}{=} 2$$

So $\boxed{\lim_{x \rightarrow 1} f(x) = 2}$ since RHL = LHL.

And since that's $f(1)$ also, that means f is continuous at $x = 1$

$\lim_{x \rightarrow 0} f(x)$: we check both sides:

$$\text{LHL: } \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} x + 2 \stackrel{\text{DSP}}{=} 2$$

$$\text{RHL: } \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} 2x^2 \stackrel{\text{DSP}}{=} 0$$

So $\boxed{\lim_{x \rightarrow 1} f(x) \text{ DNE}}$ since $\text{RHL} \neq \text{LHL}$

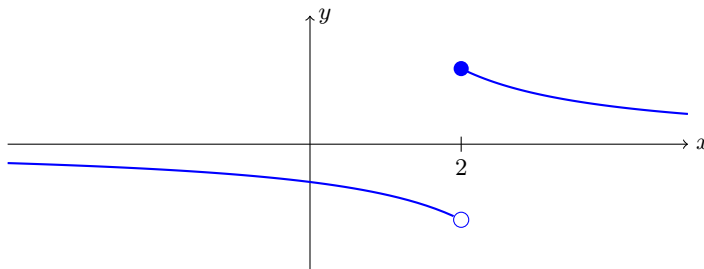
The various pieces of the function are continuous away from the break points. So the only place where f is discontinuous is at $\boxed{x = 0}$

$$54. \text{ Let } g(x) = \begin{cases} \frac{1}{x-4} & \text{if } x < 2 \\ \frac{1}{x} & \text{if } x \geq 2 \end{cases}$$

Sketch the graph. Find the numbers at which g is discontinuous. Evaluate:

$$\lim_{x \rightarrow 1} g(x) = \qquad \qquad \qquad \lim_{x \rightarrow 2} g(x) =$$

Solution. Using translation/scaling for the various pieces, and then putting them together, here's the graph:



For the limits:

$$\lim_{x \rightarrow 1} g(x) = \lim_{x \rightarrow 1} \frac{1}{x-4} = \boxed{-\frac{1}{3}}$$

$\lim_{x \rightarrow 2} g(x)$: we check both sides:

$$\text{LHL: } \lim_{x \rightarrow 2^-} g(x) = \lim_{x \rightarrow 2^-} \frac{1}{x-4} \stackrel{\text{DSP}}{=} -\frac{1}{2}$$

$$\text{RHL: } \lim_{x \rightarrow 2^+} g(x) = \lim_{x \rightarrow 2^+} \frac{1}{x} \stackrel{\text{DSP}}{=} \frac{1}{2}$$

So $\boxed{\lim_{x \rightarrow 2} g(x) \text{ DNE}}$ since $\text{RHL} \neq \text{LHL}$

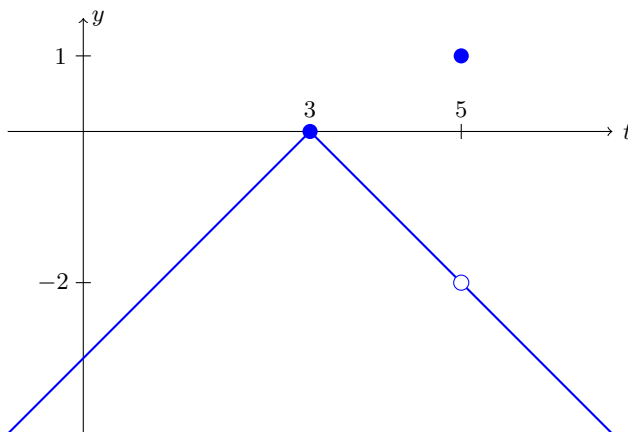
The various pieces of the function are continuous away from the break points. (Note that $1/x$ is discontinuous at $x = 0$, but g isn't given by that formula there. Similarly for $1/(x-4)$ at $x = 4$.) So the only place where g is discontinuous is at $\boxed{x = 2}$

$$55. \text{ Let } f(t) = \begin{cases} t-3 & \text{if } t \leq 3 \\ 3-t & \text{if } 3 < t < 5 \\ 1 & \text{if } t = 5 \\ 3-t & \text{if } t > 5 \end{cases}$$

Sketch the graph. Find the numbers at which f is discontinuous. Evaluate:

$$\lim_{t \rightarrow 3} f(t) = \quad \lim_{t \rightarrow 0} f(t) = \quad \lim_{t \rightarrow 5} f(t) =$$

Solution. Using translation/scaling for the various pieces, and then putting them together, here's the graph:



For the limits:

$\lim_{t \rightarrow 3} f(t)$: we check both sides:

$$\text{LHL: } \lim_{t \rightarrow 3^-} f(t) = \lim_{t \rightarrow 3^-} t - 3 \stackrel{\text{DSP}}{=} 0$$

$$\text{RHL: } \lim_{t \rightarrow 3^+} f(t) = \lim_{t \rightarrow 3^+} 3 - t \stackrel{\text{DSP}}{=} 0$$

So $\boxed{\lim_{t \rightarrow 3} f(t) = 0}$ since RHL = LHL. And since that's $f(3)$ also, that means f is continuous at $t = 3$.

$$\lim_{t \rightarrow 0} f(t) = \lim_{t \rightarrow 0} t - 3 = \boxed{-3}$$

$$\lim_{t \rightarrow 5} f(t) = \lim_{t \rightarrow 5} 3 - t = \boxed{-2}$$

However, that is **not** equal to $f(5) = -2$, so f is discontinuous at $t = 5$.

The various pieces of the function are continuous away from the break points.

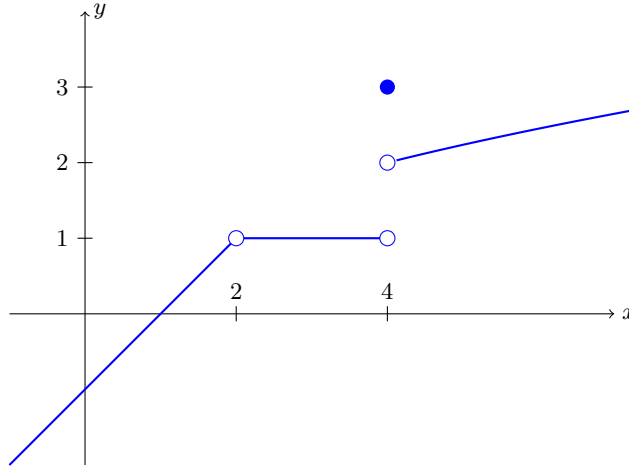
So the only place where f is discontinuous is at $\boxed{t = 5}$

$$56. \text{ Let } H(x) = \begin{cases} x - 1 & \text{if } x < 2 \\ 1 & \text{if } 2 < x < 4 \\ 3 & \text{if } x = 4 \\ \sqrt{x} & \text{if } x > 4 \end{cases}$$

Sketch the graph. Find the numbers at which H is discontinuous. Evaluate:

$$\lim_{x \rightarrow 0} H(x) = \quad \lim_{x \rightarrow 2} H(x) = \quad \lim_{x \rightarrow 4} H(x) = \quad H(4) =$$

Solution. Using translation/scaling for the various pieces, and then putting them together, here's the graph:



For the limits/value:

$$\lim_{x \rightarrow 0} H(x) = \lim_{x \rightarrow 0} x - 1 \stackrel{\text{DSP}}{=} \boxed{-1}$$

$\lim_{x \rightarrow 2} H(x)$: we check both sides:

$$\text{LHL: } \lim_{x \rightarrow 2^-} H(x) = \lim_{x \rightarrow 2^-} x - 1 \stackrel{\text{DSP}}{=} 1$$

$$\text{RHL: } \lim_{x \rightarrow 2^+} H(x) = \lim_{x \rightarrow 2^+} 1 \stackrel{\text{DSP}}{=} 1$$

So $\boxed{\lim_{x \rightarrow 2} H(x) = 1}$ since RHL = LHL. But $H(2)$ is undefined, so H is discontinuous at $x = 2$.

$\lim_{x \rightarrow 4} H(x)$: we check both sides:

$$\text{LHL: } \lim_{x \rightarrow 4^-} H(x) = \lim_{x \rightarrow 4^-} 1 \stackrel{\text{DSP}}{=} 1$$

$$\text{RHL: } \lim_{x \rightarrow 4^+} H(x) = \lim_{x \rightarrow 4^+} \sqrt{x} \stackrel{\text{DSP}}{=} 2$$

So $\boxed{\lim_{x \rightarrow 4} H(x) \text{ DNE}}$ since RHL \neq LHL.

From the formula, $\boxed{H(4) = 3}$

The various pieces of the function are continuous away from the break points.

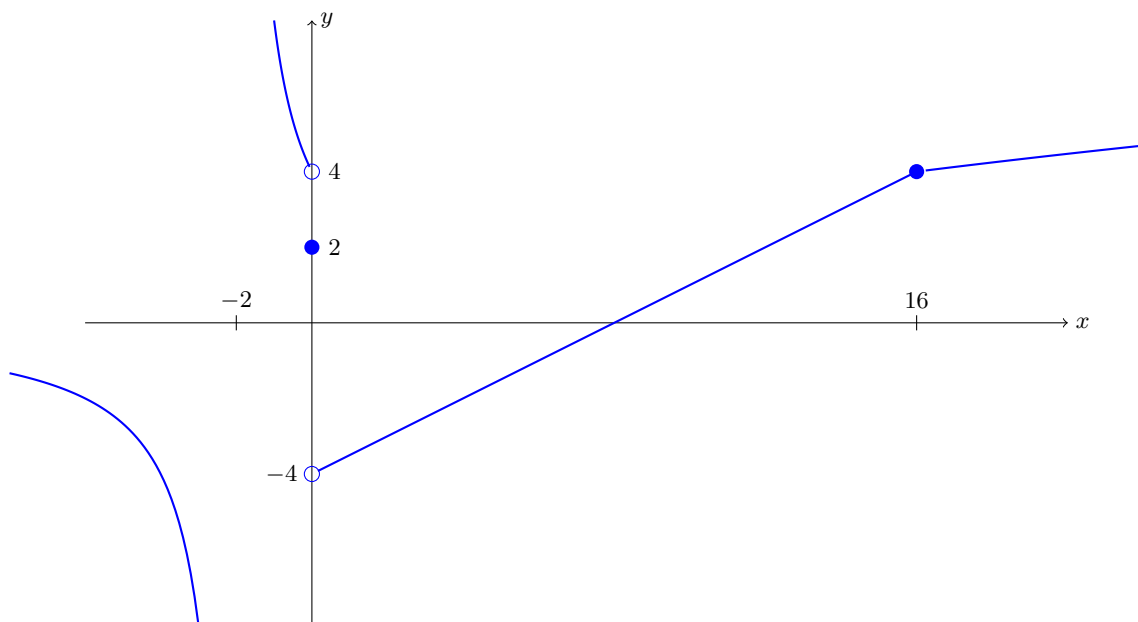
So the only places where H is discontinuous are at $\boxed{x = 2, 4}$

$$57. \text{ Let } h(x) = \begin{cases} \frac{8}{x+2} & \text{if } x < 0 \\ 2 & \text{if } x = 0 \\ \frac{1}{2}x - 4 & \text{if } 0 < x \leq 16 \\ \sqrt{x} & \text{if } x > 16 \end{cases}$$

Sketch the graph. Find the numbers at which h is discontinuous. Evaluate:

$$\lim_{x \rightarrow -2} h(x) = \qquad \lim_{x \rightarrow 0} h(x) = \qquad \lim_{x \rightarrow 16} h(x) =$$

Solution. Using translation/scaling for the various pieces, and then putting them together, here's the graph:



(To the left of $x = 16$, that is actually a (very slightly) curved graph; it's just that $y = \sqrt{x}$ gets pretty flat out there.)

For the limits:

$\lim_{x \rightarrow -2} h(x) = \frac{8}{0}$, so we check both sides:

$$\text{LHL: } \lim_{x \rightarrow -2^-} h(x) = \lim_{x \rightarrow -2^-} \frac{8}{x+2} = \frac{8}{0^-} = -\infty$$

$$\text{RHL: } \lim_{x \rightarrow -2^+} h(x) = \lim_{x \rightarrow -2^+} \frac{8}{x+2} = \frac{8}{0^+} = +\infty$$

So $\boxed{\lim_{x \rightarrow -2} h(x) \text{ DNE}}$ since $\text{RHL} \neq \text{LHL}$.

$\lim_{x \rightarrow 0} h(x)$: we check both sides:

$$\text{LHL: } \lim_{x \rightarrow 0^-} h(x) = \lim_{x \rightarrow 0^-} \frac{8}{x+2} = \frac{8}{2} = 4$$

$$\text{RHL: } \lim_{x \rightarrow 0^+} h(x) = \lim_{x \rightarrow 0^+} \frac{1}{2}x - 4 \stackrel{\text{DSP}}{=} -4$$

So $\boxed{\lim_{x \rightarrow 0} h(x) \text{ DNE}}$ since $\text{RHL} \neq \text{LHL}$.

$\lim_{x \rightarrow 16} h(x)$: we check both sides:

$$\text{LHL: } \lim_{x \rightarrow 16^-} h(x) = \lim_{x \rightarrow 16^-} \frac{1}{2}x - 4 \stackrel{\text{DSP}}{=} 8 - 4 = 4$$

$$\text{RHL: } \lim_{x \rightarrow 16^+} h(x) = \lim_{x \rightarrow 16^+} \sqrt{x} \stackrel{\text{DSP}}{=} \sqrt{16} = 4$$

So $\boxed{\lim_{x \rightarrow 16} h(x) = 4}$ since $\text{RHL} = \text{LHL}$. And since $h(16) = 4$ also, that means h is continuous at $x = 16$.

The various pieces of the function are continuous away from the breakpoints and the point $x = -2$ where the first piece is undefined.

So the only places where $h(x)$ is discontinuous are at $\boxed{x = 0, 16}$