Math 121 Final Exam   Answer Key   December 20, 2015

1. [15 Points] Evaluate each of the following **limits**. Please justify your answers. Be clear if the limit equals a value, $+\infty$ or $-\infty$, or Does Not Exist.

(a) $$\lim_{x\to\ln 3} \frac{3-e^x}{e^{-2x}-\frac{1}{9}} \overset{\left(\frac{0}{0}\right)}{} \overset{\text{L'H}}{=} \lim_{x\to\ln 3} \frac{-e^x}{-2e^{-2x}} = \frac{-e^{\ln 3}}{-2e^{-2\ln 3}}$$

$$= \frac{-3}{-2e^{\ln 3^{-2}}} = \frac{-3}{-\frac{2}{9}} = \boxed{\frac{27}{2}}$$

(b) $$\lim_{x\to 0} \frac{\ln(1-x)+\arctan x}{xe^x-\sinh x} \overset{\frac{0}{0}}{} \overset{\text{L'H}}{=} \lim_{x\to 0} \frac{-\frac{1}{1-x}+\frac{1}{1+x^2}}{xe^x+e^x-\cosh x} \overset{\frac{0}{0}}{}$$

$$\overset{\text{L'H}}{=} \lim_{x\to 0} \frac{-\frac{1}{(1-x)^2}-\frac{2x}{(1+x^2)^2}}{xe^x+e^x+e^x-\sinh x} = \boxed{-\frac{1}{2}}$$

(c) $$\lim_{x\to\infty} \left(1-\arcsin\left(\frac{6}{x}\right)\right)^x \overset{1^\infty}{} = \lim_{x\to\infty} e^{\ln\left(\left(1-\arcsin\left(\frac{6}{x}\right)\right)^x\right)}$$

$$= e^{\lim_{x\to\infty}\ln\left(\left(1-\arcsin\left(\frac{6}{x}\right)\right)^x\right)} = e^{\lim_{x\to\infty}\ln\left(\left(1-\arcsin\left(\frac{6}{x}\right)^x\right)\right)}$$

$$= e^{\lim_{x\to\infty} x\ln\left(1-\arcsin\left(\frac{6}{x}\right)\right)^{\infty\cdot 0}}$$

$$= e^{\lim_{x\to\infty} \frac{\ln\left(1-\arcsin\left(\frac{6}{x}\right)\right)}{\frac{1}{x}}\left(\frac{0}{0}\right)} \overset{\text{L'H}}{=} e^{\lim_{x\to\infty} \frac{\left(-\frac{1}{1-\arcsin\left(\frac{6}{x}\right)}\right)\left(\frac{1}{\sqrt{1-\left(\frac{6}{x}\right)^2}}\right)\left(-\frac{6}{x^2}\right)}{-\frac{1}{x^2}}}$$

$$= e^{\lim_{x\to\infty}\left(-\frac{1}{1-\arcsin\left(\frac{6}{x}\right)}\right)\left(\frac{1}{\sqrt{1-\left(\frac{6}{x}\right)^2}}\right)(6)} = \boxed{e^{-6}}$$

2. [30 Points] Evaluate each of the following **integrals**.

(a) $\displaystyle\int \frac{x^5}{\sqrt{4-x^2}}\ dx$ (using a trigonometric substitution)

$$= \int \frac{32\sin^5\theta}{\sqrt{4-4\sin^2\theta}}\ 2\cos\theta d\theta = \int \frac{32\sin^5\theta}{\sqrt{4\cos^2\theta}}\ 2\cos\theta d\theta$$

$$= 32\int \sin^5\theta\ d\theta = 32\int \sin^4\theta\sin\theta\ d\theta = 32\int \left(\sin^2\theta\right)^2\sin\theta\ d\theta$$

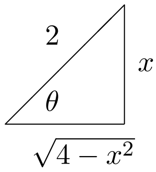
$$= 32\int \left(1-\cos^2\theta\right)^2\sin\theta\ d\theta = -32\int \left(1-u^2\right)^2\ du$$

$$= -32\int 1-2u^2+u^4\ du = -32\left(u-\frac{2}{3}u^3+\frac{1}{5}u^5\right)+C$$

$$= -32\left(\cos\theta-\frac{2}{3}\cos^3\theta+\frac{1}{5}\cos^5\theta\right)+C$$

$$= \boxed{-32\left(\frac{\sqrt{4-x^2}}{2}-\frac{2}{3}\left(\frac{\sqrt{4-x^2}}{2}\right)^3+\frac{1}{5}\left(\frac{\sqrt{4-x^2}}{2}\right)^5\right)+C}$$

Trig. Substitute $\boxed{\begin{array}{l} x = 2\sin\theta \\ \\ dx = 2\cos\theta d\theta \end{array}}$



$$\boxed{\begin{array}{l} u = \cos x \\ \\ du = -\sin x dx \\ -du = \sin x dx \end{array}}$$

(b) $\displaystyle\int_1^3 \frac{1}{\sqrt{x}\ (x+3)}\ dx = \int_1^3 \frac{1}{\sqrt{x}\ ((\sqrt{x})^2+3)}\ dx$

$$= 2\int_1^{\sqrt{3}} \frac{1}{w^2+3}\ dw = \frac{2}{\sqrt{3}}\arctan\left(\frac{w}{\sqrt{3}}\right)\Bigg|_1^{\sqrt{3}}$$

$$= \frac{2}{\sqrt{3}}\left(\arctan\left(\frac{\sqrt{3}}{\sqrt{3}}\right)-\arctan\left(\frac{1}{\sqrt{3}}\right)\right) = \frac{2}{\sqrt{3}}\left(\frac{\pi}{4}-\frac{\pi}{6}\right) = \frac{2}{\sqrt{3}}\left(\frac{\pi}{12}\right)$$

$$= \boxed{\frac{\pi}{6\sqrt{3}}}$$

Substitute $\boxed{\begin{array}{l} w = \sqrt{x} \\ \\ dw = \dfrac{1}{2\sqrt{x}}dx \end{array}}$ $\boxed{\begin{array}{l} x = 1 \Rightarrow w = 1 \\ \\ x = 3 \Rightarrow w = \sqrt{3} \end{array}}$

(c) $\displaystyle\int_{e}^{e^{\sqrt{5}}} \frac{1}{x(4+(\ln x)^2)^{\frac{3}{2}}}\ dx = \int_{1}^{\sqrt{5}} \frac{1}{(4+u^2)^{\frac{3}{2}}}\ du$

$$= \int_{u=1}^{u=\sqrt{5}} \frac{1}{(4+4\tan^2\theta)^{\frac{3}{2}}}\ 2\sec^2\theta\ d\theta = \int_{u=1}^{u=\sqrt{5}} \frac{1}{(4\sec^2\theta)^{\frac{3}{2}}}\ 2\sec^2\theta\ d\theta$$

$$= \int_{u=1}^{u=\sqrt{5}} \frac{1}{(2\sec\theta)^3}\ 2\sec^2\theta\ d\theta = \frac{1}{4}\int_{u=1}^{u=\sqrt{5}} \frac{1}{\sec\theta}\ d\theta$$

$$= \frac{1}{4}\int_{u=1}^{u=\sqrt{5}} \cos\theta\ d\theta = \frac{1}{4}\sin\theta\Big|_{u=1}^{u=\sqrt{5}}$$

$$= \frac{1}{4}\left(\frac{u}{\sqrt{u^2+4}}\right)\Bigg|_{1}^{\sqrt{5}} = \frac{1}{4}\left(\frac{\sqrt{5}}{3} - \frac{1}{\sqrt{5}}\right)$$

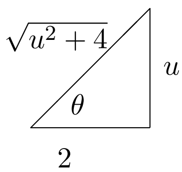
$$= \frac{1}{4}\left(\frac{\sqrt{5}}{3} - \frac{\sqrt{5}}{5}\right) = \frac{1}{4}\left(\frac{5\sqrt{5}}{15} - \frac{3\sqrt{5}}{15}\right) = \frac{1}{4}\left(\frac{2\sqrt{5}}{15}\right)$$

$$= \boxed{\frac{\sqrt{5}}{30}}$$

Standard u substitution to simplify at the start:

$$\boxed{\begin{array}{l} u = \ln x \\ \\ du = \frac{1}{x}dx \end{array}} \qquad \boxed{\begin{array}{l} x = e \Rightarrow u = 1 \\ \\ x = e^{\sqrt{5}} \Rightarrow u = \sqrt{5} \end{array}}$$

Trig. Substitute $\boxed{\begin{array}{l} u = 2\tan\theta \\ \\ du = 2\sec^2\theta d\theta \end{array}}$



(d) $\displaystyle\int x\arcsin x\ dx = \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}\ dx = \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}} \cdot \cos\theta\ d\theta$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{\sin^2\theta}{\sqrt{\cos^2\theta}}\cdot\cos\theta\ d\theta = \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{\sin^2\theta}{\cos\theta}\cdot\cos\theta\ d\theta = \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \sin^2\theta d\theta$$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{1-\cos(2\theta)}{2}d\theta = \frac{x^2}{2}\arcsin x - \frac{1}{4}\int 1-\cos(2\theta)d\theta$$

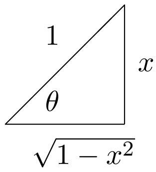
$$= \frac{x^2}{2}\arcsin x - \frac{1}{4}\left[\theta - \frac{1}{2}\sin(2\theta)\right] + C = \frac{x^2}{2}\arcsin x - \frac{1}{4}\theta + \frac{1}{8}\sin(2\theta) + C$$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{4}\theta + \frac{1}{8}2\sin\theta\cos\theta + C = \boxed{\frac{x^2}{2}\arcsin x - \frac{1}{4}\arcsin x + \frac{1}{4}x\sqrt{1-x^2} + C}$$

$$u = \arcsin x \qquad dv = x\,dx$$

$$du = \frac{1}{\sqrt{1-x^2}}dx \qquad v = \frac{x^2}{2}$$

Trig. Substitute $\boxed{x = \sin\theta \quad dx = \cos\theta\, d\theta}$



3. [24 Points] For each of the following **improper integrals**, determine whether it converges or diverges. If it converges, find its value.

(a) $\displaystyle\int_1^2 \frac{4}{x^2-8x+12}\,dx = \int_1^2 \frac{4}{(x-6)(x-2)}\,dx = \lim_{t\to 2^-}\int_1^t \frac{4}{(x-6)(x-2)}\,dx$

$\displaystyle= \lim_{t\to 2^-} \ln|x-6| - \ln|x-2|\Big|_1^t = \lim_{t\to 2^-} \ln|t-6| - \ln|t-2| - (\ln|-5| - \ln|-1|) = \lim_{t\to 2^-} \ln 4 - \ln|t-2| - \ln 5 = \ln 4 - (-\infty) - \ln 5 = \boxed{+\infty}$ Diverges

(b) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2-8x+19}\,dx = \int_0^{\infty} \frac{1}{x^2-8x+19}\,dx + \int_{-\infty}^{0} \frac{1}{x^2-8x+19}\,dx$

$\displaystyle\lim_{t\to\infty}\int_0^t \frac{1}{x^2-8x+19}\,dx + \lim_{s\to-\infty}\int_s^0 \frac{1}{x^2-8x+19}\,dx$

$\displaystyle= \lim_{t\to\infty}\int_0^t \frac{1}{(x-4)^2+3}\,dx + \lim_{s\to-\infty}\int_s^0 \frac{1}{(x-4)^2+3}\,dx$ complete the square

note: do u-sub here if needed. In that case, make sure to change your limits of integration.

$\displaystyle= \lim_{t\to\infty}\frac{1}{\sqrt{3}}\arctan\left(\frac{x-4}{\sqrt{3}}\right)\Big|_0^t + \lim_{s\to-\infty}\frac{1}{\sqrt{3}}\arctan\left(\frac{x-4}{\sqrt{3}}\right)\Big|_s^0$

$\displaystyle\lim_{t\to\infty}\frac{1}{\sqrt{3}}\left(\arctan\left(\frac{t-4}{\sqrt{3}}\right) - \arctan\left(\frac{-4}{\sqrt{3}}\right)\right) + \lim_{s\to-\infty}\frac{1}{\sqrt{3}}\left(\arctan\left(\frac{-4}{\sqrt{3}}\right) - \arctan\left(\frac{s-4}{\sqrt{3}}\right)\right)$

$\displaystyle= \frac{1}{\sqrt{3}}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right)$

$\displaystyle= \boxed{\frac{\pi}{\sqrt{3}}}$ Converges

Partial Fractions Decomposition:

$\displaystyle\frac{4}{(x-6)(x-2)} = \frac{A}{x-6} + \frac{B}{x-2}$

Clearing the denominator yields:

$4 = A(x-2) + B(x-6)$
$4 = (A+B)x - 2A - 6B$
so that $A + B = 0$, and $-2A - 6B = 4$
Solve for $A = 1$, and $B = -1$

(c) $\displaystyle\int_0^1 \frac{\ln x}{\sqrt{x}}\, dx = \int_0^1 x^{-\frac{1}{2}} \ln x\, dx = \lim_{t\to 0^+} \int_t^1 x^{-\frac{1}{2}} \ln x\, dx$

$$= \lim_{t\to 0^+} 2\sqrt{x}\ln x\Big|_t^1 - 2\int_t^1 x^{-\frac{1}{2}}\, dx$$

$$= \lim_{t\to 0^+} 2\sqrt{x}\ln x\Big|_t^1 - 4\sqrt{x}\Big|_t^1 = \lim_{t\to 0^+} 2\ln 1 - 2\sqrt{t}\ln t - 4\left(1 - \sqrt{t}\right)$$

$$= \lim_{t\to 0^+} 0 - 2\sqrt{t}\ln t - 4 \overset{(*)}{=} 0 - 4 = \boxed{-4} \text{ Converges}$$

$$(*) \quad \lim_{x\to 0^+} \sqrt{x}\ln x^{0\cdot -\infty} = \lim_{x\to 0^+} \frac{\ln x}{x^{-\frac{1}{2}}}$$

$$\lim_{x\to 0^+} \frac{\ln x}{x^{-\frac{1}{2}}}^{\frac{-\infty}{\infty}} \overset{\text{L'H}}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{2x^{\frac{3}{2}}}}$$

$$= \lim_{x\to 0^+} -2\sqrt{x} = 0$$

Integration By Parts:

$u = \ln x$	$dv = x^{-\frac{1}{2}} dx$
$du = \frac{1}{x} dx$	$v = 2\sqrt{x}$

4. [18 Points] Find the **sum** of each of the following series (which do converge):

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\, 4^{2n+1}}{3^{3n-1}} = -\frac{4^3}{3^2} + \frac{4^5}{3^5} - \frac{4^7}{3^8} + \ldots$

Here we have a geometric series with $a = -\dfrac{64}{9}$ and $r = -\dfrac{4^2}{3^3} = -\dfrac{16}{27}$

As a result, the sum is given by $\dfrac{a}{1-r} = \dfrac{-\frac{64}{9}}{1 - \left(-\frac{16}{27}\right)} = \dfrac{-\frac{64}{9}}{\frac{43}{27}} = -\dfrac{64}{9}\cdot\dfrac{27}{43} = \boxed{-\dfrac{192}{43}}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}\, 2^{n+1}\, (\ln 6)^n}{n!} = -2\sum_{n=0}^{\infty} \frac{(-1)^n\, 2^n\, (\ln 6)^n}{n!} = -2\sum_{n=0}^{\infty} \frac{(-2\ln 6)^n}{n!} = -2e^{-2\ln 6} = -2e^{\ln(6^{-2})}$

$$= -\frac{2}{36} = \boxed{-\frac{1}{18}}$$

(c) $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{2^{4n}(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{4^{2n}(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{4}\right)^{2n}}{(2n)!} = \cos\left(\frac{\pi}{4}\right) = \boxed{\frac{\sqrt{2}}{2}}$

(d) $-\frac{1}{5} + \frac{1}{2 \cdot 5^2} - \frac{1}{3 \cdot 5^3} + \frac{1}{4 \cdot 5^4} - \ldots = -\left(\frac{1}{5} - \frac{1}{2 \cdot 5^2} + \frac{1}{3 \cdot 5^3} - \frac{1}{4 \cdot 5^4} + \ldots\right)$

$= -\ln\left(1 + \frac{1}{5}\right) = \boxed{-\ln\left(\frac{6}{5}\right)}$

(e) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots = \arctan(1) = \boxed{\frac{\pi}{4}}$

(f) $\sum_{n=0}^{\infty} \frac{1}{e^n} = 1 + \frac{1}{e} + \frac{1}{e^2} + \ldots$

This is a geometric series with $a = 1$ and $r = \frac{1}{e}$ and SUM$=\frac{a}{1-r} = \frac{1}{1 - \frac{1}{e}} = \frac{1}{\left(\frac{e-1}{e}\right)} = \boxed{\frac{e}{e-1}}$

(g) $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{(36)^n\ (2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n}\ (2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{6}\right)^{2n}}{(2n+1)!} \left(\frac{\frac{\pi}{6}}{\frac{\pi}{6}}\right)$

$= \frac{6}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{6}\right)^{2n+1}}{(2n+1)!} = \frac{6}{\pi} \sin\left(\frac{\pi}{6}\right) = \frac{6}{\pi}\left(\frac{1}{2}\right) = \boxed{\frac{3}{\pi}}$

5. [35 Points] In each case determine whether the given series is **absolutely convergent**, **conditionally convergent**, or **divergent**. Justify your answers.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n\ (n^4 + 7)}{n^7 + 4}$

First examine the absolute series $\sum_{n=1}^{\infty} \frac{n^4 + 7}{n^7 + 4} \approx \sum_{n=1}^{\infty} \frac{n^4}{n^7} = \sum_{n=1}^{\infty} \frac{1}{n^3}$

which is a convergent p-series with $p = 3 > 1$.

Next check

Check: $\lim_{n\to\infty} \frac{\frac{n^4+7}{n^7+4}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{n^7 + 7n^3}{n^7 + 4} = \lim_{n\to\infty} \frac{1 + \frac{7}{n^4}}{1 + \frac{4}{n^7}} = 1$ which is finite and non-zero ($0 < 1 < \infty$).

Therefore, these two series share the same behavior, and the absolute series $\sum_{n=1}^{\infty} \frac{n^4+7}{n^7+4}$ is also convergent by Limit Comparison Test (LCT). (Note: the Original Series is Convergent by ACT.) Finally, we have $\boxed{\text{Absolute Convergence}}$.

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n \arctan(7n)}{e^n+7}$

First we analyze the absolute series $\sum_{n=1}^{\infty} \frac{\arctan(7n)}{e^n+7}$

We can bound the terms here:
$\frac{\arctan(7n)}{e^n+7} < \frac{\frac{\pi}{2}}{e^n+7} < \frac{\frac{\pi}{2}}{e^n}$.
Note that
$\sum_{n=1}^{\infty} \frac{\frac{\pi}{2}}{e^n} = \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{e^n}$ is a constant multiple of a convergent Geometric series with $|r| = \frac{1}{e} < 1$ and therefore, is convergent.

Therefore the absolute series converges by CT. (Note: The original series converges by ACT.) Finally, we have $\boxed{\text{Absolute Convergence}}$.

(c) $\sum_{n=1}^{\infty} n \cdot \arctan\left(\frac{1}{n}\right)$

$\boxed{\text{Diverges}}$ by the n^{th} term Divergence Test since

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n \cdot \arctan\left(\frac{1}{n}\right) \overset{\infty\cdot 0}{=} \lim_{x\to\infty} x \cdot \arctan\left(\frac{1}{x}\right)$$

$$= \lim_{x\to\infty} \frac{\arctan\left(\frac{1}{x}\right)}{\frac{1}{x}} \overset{\frac{0}{0}}{\underset{\text{L'H}}{=}} \lim_{x\to\infty} \frac{\frac{1}{1+\left(\frac{1}{x}\right)^2}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{1}{1+\frac{1}{x^2}} = 1 \neq 0$$

(d) $\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{n+3}$

First, we show the absolute series is divergent. Note that $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+3} \approx \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which is a divergent p-series with $p = \frac{1}{2} < 1$. Next, Check: $\lim_{n\to\infty} \dfrac{\dfrac{\sqrt{n}}{n+3}}{\dfrac{1}{\sqrt{n}}} = \lim_{n\to\infty} \dfrac{n}{n+3} \dfrac{\left(\dfrac{1}{n}\right)}{\left(\dfrac{1}{n}\right)} = \lim_{n\to\infty} \dfrac{1}{1+\dfrac{3}{n}} = 1$ which is finite and non-zero. Therefore, these two series share the same behavior. Since $\sum_{n=1}^{\infty} \dfrac{1}{\sqrt{n}}$ is the divergent p-Series $\left(p = \dfrac{1}{2} < 1\right)$, then the absolute series $\sum_{n=1}^{\infty} \dfrac{\sqrt{n}}{n+3}$ is also divergent by Limit Comparison Test. As a result, we have no chance for Absolute Convergence.

Secondly, we are left to examine the original alternating series with the Alternating Series Test.

- $b_n = \dfrac{\sqrt{n}}{n+3} > 0$

- $\lim_{n\to\infty} \dfrac{\sqrt{n}}{n+3} = \lim_{n\to\infty} \dfrac{\sqrt{n}}{n+3} \cdot \dfrac{\dfrac{1}{n}}{\dfrac{1}{n}} = \lim_{n\to\infty} \dfrac{\frac{1}{\sqrt{n}}}{1+\frac{3}{n}} = 0$

- $b_{n+1} < b_n$ because $f(x) = \dfrac{\sqrt{x}}{x+3}$ has $f'(x) = \dfrac{3-x}{2\sqrt{x}(x+3)^2} < 0$ when $x > 3$

Therefore, the original series converges by the Alternating Series Test. Finally, we can conclude the original series is $\boxed{\text{Conditionally Convergent}}$

(e) $\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}\ e^{3n}\ (3n)!}{n^n\ 4^{2n}\ (n!)^2}$

Try Ratio Test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\dfrac{(-1)^{n+2}e^{3n+3}(3(n+1))!}{(n+1)^{n+1}4^{2n+2}((n+1)!)^2}}{\dfrac{(-1)^{n+1}e^{3n}(3n)!}{n^n4^{2n}(n!)^2}} \right|$$

$$= \lim_{n\to\infty} \frac{e^{3n+3}}{e^{3n}} \cdot \frac{(3n+3)!}{(3n)!} \cdot \frac{n^n}{(n+1)^{n+1}} \cdot \frac{4^{2n}}{4^{2n+2}} \cdot \frac{(n!)^2}{((n+1)!)^2}$$

$$= \lim_{n\to\infty} \frac{e^{3n}e^3}{e^{3n}} \cdot \frac{(3n+3)(3n+2)(3n+1)(3n)!}{(3n)!} \cdot \frac{n^n}{(n+1)^n(n+1)} \cdot \frac{4^{2n}}{4^{2n}4^2} \cdot \frac{(n!)^2}{(n+1)^2(n!)^2}$$

$$= \lim_{n\to\infty} \frac{e^3}{1} \cdot \frac{3(n+1)(3n+2)(3n+1)}{1} \cdot \frac{n^n}{(n+1)^n} \cdot \frac{1}{n+1} \cdot \frac{1}{16} \cdot \frac{1}{(n+1)^2}$$

$$= \lim_{n\to\infty} \frac{e^3}{1} \cdot \frac{3(3n+2)(3n+1)}{1} \cdot \frac{1}{e} \cdot \frac{1}{16} \cdot \frac{1}{(n+1)^2}$$

$$= \lim_{n\to\infty} \frac{e^3}{1} \cdot \frac{3(9n^2+9n+2}{1} \cdot \frac{1}{e} \cdot \frac{1}{16} \cdot \frac{1}{n^2+2n+1}$$

$$= \lim_{n\to\infty} \frac{27e^2}{16} > 1$$

Therefore, the series $\boxed{\text{Diverges}}$ by the Ratio Test.

6. [15 Points] Find the **Interval** and **Radius** of Convergence for the following power series. Analyze carefully and with full justification.

$$\sum_{n=1}^{\infty} \frac{(-1)^n \ (\ln n) \ (4x-1)^n}{n^2 \cdot 5^n}$$

Use Ratio Test.

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\dfrac{(-1)^{n+1} \ln(n+1)(4x-1)^{n+1}}{(n+1)^2 5^{n+1}}}{\dfrac{(-1)^n \ln n (4x-1)^n}{(n+1)5^n}} \right|$$

$$= \lim_{n\to\infty} \frac{\ln(n+1)}{\ln n} \frac{|4x-1|}{5} \overset{(**)}{=} \frac{|4x-1|}{5}$$

The Ratio Test gives convergence for x when $\dfrac{|4x-1|}{5} < 1$ or $|4x-1| < 5$.

That is $-5 < 4x - 1 < 5 \implies -4 < 4x < 6 \implies -1 < x < \dfrac{3}{2}$

$$(**) \lim_{n\to\infty} \frac{\ln(n+1)}{\ln n} \overset{\frac{\infty}{\infty}}{} = \lim_{x\to\infty} \frac{\ln(x+1)}{\ln x} \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{\frac{1}{x+1}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{x}{x+1} = \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}}$$

Endpoints:

•$x = -1$ The original series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \ (\ln n) \ (-5)^n}{n^2 \cdot 5^n} = \sum_{n=1}^{\infty} \frac{(\ln n) 5^n}{n^2 5^n} = \sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

We bound $\dfrac{\ln n}{n^2} < \dfrac{\sqrt{n}}{n^2} = \dfrac{1}{n^{\frac{3}{2}}}$ and

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ is a convergent p-series with $p = \dfrac{3}{2} > 1$. Then our endpoint series converges by CT.

•$x = \dfrac{3}{2}$ The original series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \ (\ln n) \ (5)^n}{n^2 \cdot 5^n} = \sum_{n=1}^{\infty} \frac{(-1)^n (\ln n)}{n^2}$

Here we consider the absolute series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$.

It is convergent as shown directly above. Therefore this alternating endpoint series is convergent by ACT.

Finally, Interval of Convergence $\boxed{I = \left[-1, \dfrac{3}{2}\right]}$ with Radius of Convergence $\boxed{R = \dfrac{5}{4}}$.

7. [8 Points]

(a) Write the MacLaurin Series for the hyperbolic cosine $f(x) = \cosh x$.

$f(x) = \cosh x$

$$\cosh x = \frac{e^x + e^{-x}}{2} = \frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{x^n}{n!} + \sum_{n=0}^{\infty}\frac{(-x)^n}{n!}\right) = \frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{x^n}{n!} + \sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!}\right)$$

$$= \frac{1}{2}\left(\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots\right)\right)\right)$$

$$= \frac{1}{2}\left(2 + 2\left(\frac{x^2}{2!}\right) + 2\left(\frac{x^4}{4!}\right) + \ldots\right)$$

$$= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots = \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$$ Here $R = \infty$ because $\sinh x$ is built of exponentials.

OR you could use the manual chart method:

$f(x) = \cosh x \quad f(0) = \cosh 0 = 1$

$f'(x) = \sinh x \quad f'(0) = \sinh 0 = 0$

$f''(x) = \cosh x \quad f''(0) = \cosh 0 = 1$

$f'''(x) = \sinh x \quad f'''(0) = \sinh 0 = 0$

$\ldots \quad \ldots$

Finally, MacLaurin Series

$$= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$$

$$= 1 + 0x + \frac{1}{2!}x^2 + \frac{0}{3!}x^3 + \frac{1}{4!}x^4 + \frac{0}{5!}x^5 + \ldots$$

$$= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots$$

$$= \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$$ Match!

Note, all of the odd powered derivatives equal 0, so we are left with only the even powered terms.

(b) Write the MacLaurin Series for $f(x) = \cosh(2x^3)$.

$$\cosh(2x^3) = \sum_{n=0}^{\infty}\frac{(2x^3)^{2n}}{(2n)!} = \boxed{\sum_{n=0}^{\infty}\frac{2^{2n}x^{6n}}{(2n)!}} = 1 + 2x^6 + \frac{2}{3}x^{12} + \ldots$$

(c) Use this series to determine the **twelfth**, and **thirteenth**, derivatives of $f(x) = \cosh(2x^3)$ evaluated at $\boldsymbol{x = 0}$. That is, compute $f^{(12)}(0)$ and $f^{13}(0)$. Do **not** simplify your answers here.

$$\frac{f^{12}(0)}{(12)!} = \frac{2}{3} \rightarrow f^{12}(0) = \boxed{\frac{2(12)!}{3}}$$

$$\frac{f^{13}(0)}{(13)!} = 0 \rightarrow f^{13}(0) = \boxed{0}$$

8. [12 Points] Please analyze with detail and justify carefully. Simplify your answers.

(a) Use the MacLaurin series representation for $\boldsymbol{f(x) = x\sin(x^2)}$ to **Estimate** $\int_0^1 x\sin\left(x^2\right)\, dx$ with error less than $\frac{1}{100}$. Justify in words that your error is less than $\frac{1}{100}$.

$$\int_0^1 x\sin\left(x^2\right)\, dx = \int_0^1 x\sum_{n=0}^{\infty}\frac{(-1)^n(x^2)^{2n+1}}{(2n+1)!}\, dx = \int_0^1 x\sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+2}}{(2n+1)!}\, dx$$

$$= \int_0^1 \sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+3}}{(2n+1)!}\, dx = \sum_{n=0}^{\infty}\left.\frac{(-1)^n x^{4n+4}}{(2n+1)!(4n+4)}\right|_0^1$$

$$= \left.\frac{x^4}{4} - \frac{x^8}{3!8} + \frac{x^{12}}{5!(12)} - \dots\right|_0^1 = \frac{1}{4} - \frac{1}{48} + \frac{1}{1440} - \dots - (0 - 0 + 0 - \dots)$$

$$\approx \frac{1}{4} - \frac{1}{48} = \frac{11}{48} \leftarrow \text{estimate}$$

Using ASET we can estimate the full sum using only the first two terms with error *at most* $\frac{1}{1440} < \frac{1}{100}$ as desired.

(b) Estimate $\cos\left(\frac{1}{2}\right)$ with error less than $\frac{1}{100}$. Justify in words that your error is indeed less than $\frac{1}{100}$.

$$\cos x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\cos\left(\frac{1}{2}\right) = 1 - \frac{(\frac{1}{2})^2}{2!} + \frac{(\frac{1}{2})^4}{4!} + \dots = 1 - \frac{1}{8} + \frac{1}{384} + \dots \approx 1 - \frac{1}{8} = \frac{7}{8} \leftarrow \text{estimate}$$

Using ASET we can estimate the full sum using only the first two terms with error *at most* $\frac{1}{384} < \frac{1}{100}$ as desired.

9. [10 Points] Consider the region bounded by $y = \cos x$, $y = x + 1$, $x = 0$ and $x = \frac{\pi}{2}$. Rotate the region about the vertical line $\boxed{x = 3}$. **COMPUTE** the **volume** of the resulting solid using the Cylindrical Shells Method. Sketch the solid, along with one of the approximating cylindrical shells.

See me for a full sketch.

$$V = \int_0^{\frac{\pi}{2}} 2\pi \text{ radius height } dx = 2\pi\int_0^{\frac{\pi}{2}}(3 - x)(x + 1 - \cos x)dx$$

$$= 2\pi \int_0^{\frac{\pi}{2}} 3x + 3 - 3\cos x - x^2 - x + x\cos x dx = 2\pi \int_0^{\frac{\pi}{2}} 2x + 3 - 3\cos x - x^2 + x\cos x dx$$

$$\overset{\text{IBP}}{=} 2\pi \left(x^2 + 3x - 3\sin x - \frac{x^3}{3} + (x\sin x + \cos x) \right) \Bigg|_0^{\frac{\pi}{2}}$$

$$= 2\pi \left(\frac{\pi^2}{4} + \frac{3\pi}{2} - 3 - \frac{\pi^3}{24} + \frac{\pi}{2} + 0 \right) - (0 + 0 - 0 - 0 + 0 + \cos 0)$$

$$= \boxed{2\pi \left(\frac{\pi^2}{4} + 2\pi - 4 - \frac{\pi^3}{24} \right)}$$

IBP

$u = x$	$dv = \cos x dx$
$du = dx$	$v = \sin x$

$$\int x\cos x \, dx = x\sin x - \int \sin x \, dx = x\sin x + \cos x + C$$

10. [18 Points]

(a) Consider the Parametric Curve represented by $x = t + \dfrac{1}{1+t}$ and $y = 2\ln(1+t)$.

COMPUTE the **arclength** of this parametric curve for $0 \le t \le 4$.

First $\dfrac{dx}{dt} = 1 - \dfrac{1}{(1+t)^2}$ and $\dfrac{dy}{dt} = \dfrac{2}{1+t}$.

$$L = \int_0^4 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt = \int_0^4 \sqrt{\left(1 - \frac{1}{(1+t)^2}\right)^2 + \left(\frac{2}{1+t}\right)^2} \, dt$$

$$\int_0^4 \sqrt{1 - \frac{2}{(1+t)^2} + \frac{1}{(1+t)^4} + \frac{4}{(1+t)^2}} \, dt = \int_0^4 \sqrt{1 + \frac{2}{(1+t)^2} + \frac{1}{(1+t)^4}} \, dt$$

$$= \int_0^4 \sqrt{\left(1 + \frac{1}{(1+t)^2}\right)^2} \, dt = \int_0^4 1 + \frac{1}{(1+t)^2} \, dt$$

$$= t - \frac{1}{1+t} \Bigg|_0^4 = 4 - \frac{1}{5} - (0 - 1) = 5 - \frac{1}{5} = \boxed{\frac{24}{5}}$$

(b) Consider a *different* Parametric Curve represented by $x = t - e^{2t}$ and $y = 1 - \sqrt{8}\, e^t$.

COMPUTE the **surface area** obtained by rotating this curve about the $\boxed{y\text{-axis}}$, for $0 \le t \le 3$.

First $\dfrac{dx}{dt} = 1 - 2e^{2t}$ and $\dfrac{dy}{dt} = -\sqrt{8}\, e^t$.

$$\text{S.A.} = \int_0^3 2\pi x(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt = 2\pi \int_0^3 (t - e^{2t}) \sqrt{(1 - 2e^{2t})^2 + \left(-\sqrt{8}\, e^t\right)^2} \, dt$$

$$= 2\pi \int_0^3 (t - e^{2t})\sqrt{1 - 4e^{2t} + 4e^{4t} + 8e^{2t}}\ dt = 2\pi \int_0^3 (t - e^{2t})\sqrt{1 + 4e^{2t} + 4e^{4t}}\ dt$$

$$= 2\pi \int_0^3 (t - e^{2t})\sqrt{(1 + 2e^{2t})^2}\ dt = 2\pi \int_0^3 (t - e^{2t})(1 + 2e^{2t})\ dt = 2\pi \int_0^3 t - e^{2t} + 2te^{2t} - 2e^{4t}\ dt$$

$$\overset{\text{IBP}}{=} 2\pi \left(\frac{t^2}{2} - \frac{e^{2t}}{2} + 2\left(\frac{te^{2t}}{2} - \frac{e^{2t}}{4} \right) - \frac{e^{4t}}{2} \right) \Bigg|_0^3$$

$$= 2\pi \left(\frac{9}{2} - \frac{e^6}{2} + 2\left(\frac{3e^6}{2} - \frac{e^6}{4} \right) - \frac{e^{12}}{2} - \left(0 - \frac{1}{2} + 0 - \frac{1}{2} - \frac{1}{2} \right) \right)$$

$$= \boxed{2\pi \left(6 + 2e^6 - \frac{e^{12}}{2} \right)}$$

IBP

$u = t$	$dv = e^{2t}dt$
$du = dt$	$v = \frac{e^{2t}}{2}$

$$\int te^{2t}\ dt = \frac{te^{2t}}{2} - \int \frac{e^{2t}}{2}\ dt = \frac{te^{2t}}{2} - \frac{e^{2t}}{4} + C$$

11. [15 Points] Compute the **area** bounded outside the polar curve $\boldsymbol{r = 1 + \sin\theta}$ and inside the polar curve $\boldsymbol{r = 3\sin\theta}$. **Sketch** the Polar curves **and** shade the bounded area.

These two polar curves intersect when

$1 + \sin\theta = 3\sin\theta \Rightarrow 2\sin\theta = 1 \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \dfrac{\pi}{6}$ or $\theta = \dfrac{5\pi}{6}$.

Using symmetry, we will integrate from $\theta = \dfrac{\pi}{6}$ to $\theta = \dfrac{\pi}{2}$ and double that area.

$$\text{Area} = A = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left((\text{outer}\ \ r)^2 - (\text{inner}\ \ r)^2 \right)\ d\theta$$

$$= 2\left(\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left((\text{outer}\ \ r)^2 - (\text{inner}\ \ r)^2 \right)\ d\theta \right)$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left((3\sin\theta)^2 - (1 + \sin\theta)^2 \right)\ d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 9\sin^2\theta - (1 + 2\sin\theta + \sin^2\theta)\ d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 9\sin^2\theta - 1 - 2\sin\theta - \sin^2\theta\ d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 8\sin^2\theta - 1 - 2\sin\theta\ d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 8\left(\frac{1 - \cos(2\theta)}{2} \right) - 1 - 2\sin\theta\ d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 4\left(1 - \cos(2\theta)\right) - 1 - 2\sin\theta\ d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 4 - 4\cos(2\theta) - 1 - 2\sin\theta\ d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 3 - 4\cos(2\theta) - 2\sin\theta \ d\theta$$

$$= 3\theta - 2\sin(2\theta) + 2\cos\theta \Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \left(3\left(\frac{\pi}{2}\right) - 2\sin\left(\frac{2\pi}{2}\right) + 2\cos\left(\frac{\pi}{2}\right)\right) - \left(3\left(\frac{\pi}{6}\right) - 2\sin\left(\frac{2\pi}{6}\right) + 2\cos\left(\frac{\pi}{6}\right)\right)$$

$$= \frac{3\pi}{2} - 2(0) + 2(0) - \left(\frac{\pi}{2} - 2\left(\frac{\sqrt{3}}{2}\right) + 2\left(\frac{\sqrt{3}}{2}\right)\right) = \boxed{\pi}$$