**Math 121 Final Exam December 18, 2016**

**1.** [18 Points] Evaluate the following **limit**. Please justify your answer. Be clear if the limit equals a value, $+\infty$ or $-\infty$, or Does Not Exist. Simplify.

(a) $$\lim_{x\to 0} \frac{3xe^x - \arctan(3x)}{x+\ln(1-x)} \left(\frac{0}{0}\right) \overset{\text{L'H}}{=} \lim_{x\to 0} \frac{3xe^x + 3e^x - \dfrac{3}{1+9x^2}}{1-\dfrac{1}{1-x}} \left(\frac{0}{0}\right)$$

$$\overset{\text{L'H}}{=} \lim_{x\to 0} \frac{3xe^x + 3e^x + 3e^x + \dfrac{54x}{(1+9x^2)^2}}{-\dfrac{1}{(1-x)^2}} = \frac{3+3}{-1} = \boxed{-6}$$

(b) Compute $\displaystyle\lim_{x\to\infty} \left(\frac{x}{x+7}\right)^{x} {}^{(1^\infty)}$

$$= \lim_{x\to\infty} e^{\ln\left(\frac{x}{x+7}\right)^x} = e^{\lim_{x\to\infty} \ln\left[\left(\frac{x}{x+7}\right)^x\right]} = e^{\lim_{x\to\infty} x\ln\left(\frac{x}{x+7}\right)} {}^{(\infty\cdot 0)}$$

$$= e^{\lim_{x\to\infty} \frac{\ln\left(\frac{x}{x+7}\right)}{\frac{1}{x}}} {}^{\frac{0}{0}} \overset{\text{L'H}}{=} e^{\lim_{x\to\infty} \frac{\left(\frac{x+7}{x}\right)\left(\frac{(x+7)(1)-x(1)}{(x+7)^2}\right)}{-\frac{1}{x^2}}} = e^{\lim_{x\to\infty} \frac{\left(\frac{x+7}{x}\right)\left(\frac{7}{(x+7)^2}\right)}{-\frac{1}{x^2}}}$$

$$= e^{\lim_{x\to\infty} \left(\frac{x+7}{x}\right)\left(\frac{7}{(x+7)^2}\right)(-x^2)} = e^{\lim_{x\to\infty} \left(\frac{-7x}{x+7}\right)} {}^{\frac{\infty}{\infty}} \overset{\text{L'H}}{=} e^{\frac{-7}{1}} = \boxed{\frac{1}{e^7}}$$

**2.** [20 Points] Evaluate each of the following **integrals**.

(a) $$\int \frac{1}{(x^2+4)^{\frac{7}{2}}}\,dx = \int \frac{1}{(4+4\tan^2\theta)^{\frac{7}{2}}}\,2\sec^2\theta\,d\theta = \int \frac{1}{(4(1+\tan^2\theta))^{\frac{7}{2}}}\,2\sec^2\theta\,d\theta$$

$$= \int \frac{1}{(4(\sec^2\theta))^{\frac{7}{2}}}\,2\sec^2\theta\,d\theta = \int \frac{1}{\left(\sqrt{4}\sqrt{\sec^2\theta}\right)^7}\,2\sec^2\theta\,d\theta$$

$$= \int \frac{1}{2^7\sec^7\theta}\,2\sec^2\theta\,d\theta = \frac{1}{2^6}\int \frac{\sec^2\theta}{\sec^7\theta}\,d\theta = \frac{1}{64}\int \frac{1}{\sec^5\theta}\,d\theta = \frac{1}{64}\int \cos^5\theta\,d\theta$$

$$= \frac{1}{64}\int \cos^4\theta\cos\theta\,d\theta = \frac{1}{64}\int \left(1-\sin^2\theta\right)^2\cos\theta\,d\theta = \frac{1}{64}\int (1-w^2)^2\,dw$$

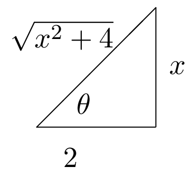
Substitute $\boxed{\begin{array}{l} w = \sin\theta \\ dw = \cos\theta d\theta \end{array}}$

$$= \frac{1}{64}\int 1 - 2w^2 + w^4 \ dw = \frac{1}{64}\left(w - \frac{2w^3}{3} + \frac{w^5}{5}\right) + C$$

$$= \frac{1}{64}\left(\sin\theta - \frac{2\sin^3\theta}{3} + \frac{\sin^5\theta}{5}\right) + C$$

$$= \boxed{\frac{1}{64}\left(\frac{x}{\sqrt{x^2+4}} - \frac{2}{3}\left(\frac{x}{\sqrt{x^2+4}}\right)^3 + \frac{1}{5}\left(\frac{x}{\sqrt{x^2+4}}\right)^5\right) + C}$$

Trig. Substitute $\boxed{\begin{array}{l} x = 2\tan\theta \\ \\ dx = 2\sec^2\theta d\theta \end{array}}$




(b) $$\int_0^1 x\arcsin x \ dx = \frac{x^2}{2}\arcsin x\Big|_0^1 - \frac{1}{2}\int_0^1 \frac{x^2}{\sqrt{1-x^2}} \ dx$$

$$= \frac{x^2}{2}\arcsin x\Big|_0^1 - \frac{1}{2}\int_{x=0}^{x=1} \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}} \cdot \cos\theta \ d\theta$$

$$= \frac{x^2}{2}\arcsin x\Big|_0^1 - \frac{1}{2}\int_{x=0}^{x=1} \frac{\sin^2\theta}{\sqrt{\cos^2\theta}} \cdot \cos\theta \ d\theta = \frac{x^2}{2}\arcsin x\Big|_0^1 - \frac{1}{2}\int_{x=0}^{x=1} \frac{\sin^2\theta}{\cos\theta} \cdot \cos\theta \ d\theta$$

$$= \frac{x^2}{2}\arcsin x\Big|_0^1 - \frac{1}{2}\int_{x=0}^{x=1} \sin^2\theta d\theta$$

$$= \frac{x^2}{2}\arcsin x\Big|_0^1 - \frac{1}{2}\int_{x=0}^{x=1} \frac{1-\cos(2\theta)}{2} d\theta = \frac{x^2}{2}\arcsin x\Big|_0^1 - \frac{1}{4}\int_{x=0}^{x=1} 1 - \cos(2\theta) d\theta$$

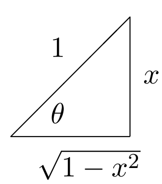
$$= \frac{x^2}{2}\arcsin x\Big|_0^1 - \frac{1}{4}\left[\theta - \frac{1}{2}\sin(2\theta)\right]\Big|_{x=0}^{x=1} = \frac{x^2}{2}\arcsin x\Big|_0^1 - \frac{1}{4}\theta + \frac{1}{8}\sin(2\theta)\Big|_{x=0}^{x=1}$$

$$= \frac{x^2}{2}\arcsin x\Big|_0^1 - \frac{1}{4}\theta + \frac{1}{8}2\sin\theta\cos\theta\Big|_{x=0}^{x=1} = \frac{x^2}{2}\arcsin x\Big|_0^1 - \frac{1}{4}\arcsin x + \frac{1}{4}x\sqrt{1-x^2}\Big|_0^1$$

$$= \frac{1}{2}\arcsin 1 - 0 - \left(\frac{1}{4}\arcsin 1 - \frac{1}{4}\arcsin 0\right) + 0 - 0 = \frac{1}{2}\left(\frac{\pi}{2}\right) - \frac{1}{4}\left(\frac{\pi}{2}\right) = \boxed{\frac{\pi}{8}}$$

$$\boxed{\begin{array}{ll} u = \arcsin x & dv = xdx \\ \\ du = \frac{1}{\sqrt{1-x^2}}dx & v = \frac{x^2}{2} \end{array}}$$

Trig. Substitute $\boxed{\begin{array}{l} x = \sin\theta \\ \\ dx = \cos\theta d\theta \end{array}}$



**3.** [30 Points] For the following **improper integral**, determine whether it converges or diverges. If it converges, find its value. Simplify.

(a) $\displaystyle\int_0^{e^4} \frac{1}{x\left[16+(\ln x)^2\right]}\,dx \lim_{t\to 0^+}\int_t^{e^4} \frac{1}{x[16+(\ln x)^2]}\,dx$

$\displaystyle= \lim_{t\to 0^+}\int_{\ln t}^{4} \frac{1}{16+u^2}\,du = \lim_{t\to 0^+}\frac{1}{4}\arctan\left(\frac{u}{4}\right)\Bigg|_{\ln t}^{4} = \lim_{t\to 0^+}\frac{1}{4}\left(\arctan\left(\frac{4}{4}\right)-\arctan\left(\frac{\ln t}{4}\right)\right)$

$\displaystyle= \lim_{t\to 0^+}\frac{1}{4}\left(\arctan(1)-\arctan\left(\frac{\ln t}{4}\right)\right) = \frac{1}{4}\left(\frac{\pi}{4}-\left(-\frac{\pi}{2}\right)\right) = \frac{1}{4}\left(\frac{\pi}{4}+\frac{2\pi}{4}\right) = \boxed{\frac{3\pi}{16}}$

Substitute $\boxed{\begin{array}{l} u=\ln x \\ du=\frac{1}{x}dx \end{array}}$ $\boxed{\begin{array}{l} x=t\Rightarrow u=\ln t \\ x=e^4\Rightarrow u=\ln e^4=4 \end{array}}$

(b) $\displaystyle\int_1^2 \frac{4}{x^2-6x+5}\,dx = \int_1^2 \frac{4}{(x-5)(x-1)}\,dx = \lim_{t\to 1^+}\int_t^2 \frac{4}{(x-5)(x-1)}\,dx$

$\displaystyle\overset{\text{PFD}}{=} \lim_{t\to 1^+}\int_t^2 \frac{1}{x-5}-\frac{1}{x-1}\,dx = \lim_{t\to 1^+}\ln|x-5|-\ln|x-1|\Bigg|_t^2$

$\displaystyle= \lim_{t\to 1^+}\ln|2-5|-\ln|2-1|-(\ln|t-5|-\ln|t-1|) = \lim_{t\to 1^+}\ln 3-0-\ln|t-5|+\ln|t-1|$

$= \ln 3-\ln 4+(-\infty) = \boxed{-\infty}$ Diverges

Partial Fractions Decomposition:

$\displaystyle\frac{4}{(x-5)(x-1)} = \frac{A}{x-5}+\frac{B}{x-1}$

Clearing the denominator yields:

$4 = A(x-1)+B(x-5)$
$4 = Ax-A+Bx-5B$
$4 = (A+B)x-A-5B$
so that $A+B=0,$ and $-A-5B=4$
So lve for $A=1,$ and $B=-1$

(c) $\displaystyle\int_4^{\infty} \frac{4}{x^2-6x+12}\,dx = 4\lim_{t\to\infty}\int_4^t \frac{1}{x^2-6x+12}\,dx = 4\lim_{t\to\infty}\int_4^t \frac{1}{(x-3)^2+3}\,dx$

$\displaystyle= 4\lim_{t\to\infty}\int_1^{t-3}\frac{1}{w^2+3}\,dw = 4\lim_{t\to\infty}\frac{1}{\sqrt{3}}\arctan\left(\frac{w}{\sqrt{3}}\right)\Bigg|_1^{t-3}$

$\displaystyle= \frac{4}{\sqrt{3}}\lim_{t\to\infty}\left(\arctan\left(\frac{t-3}{\sqrt{3}}\right)-\arctan\left(\frac{1}{\sqrt{3}}\right)\right)$

$\displaystyle= \frac{4}{\sqrt{3}}\lim_{t\to\infty}\left(\arctan\left(\frac{t-3}{\sqrt{3}}\right)-\arctan\left(\frac{1}{\sqrt{3}}\right)\right)$

$$= \frac{4}{\sqrt{3}}\left(\frac{\pi}{2} - \frac{\pi}{6}\right) = \boxed{\frac{4\pi}{3\sqrt{3}}} \quad \text{Converges}$$

Substitute $\boxed{\begin{array}{l} w = x - 3 \\ \\ dw = dx \end{array}}$ $\boxed{\begin{array}{l} x = 4 \Rightarrow w = 1 \\ \\ x = t \Rightarrow w = t - 3 \end{array}}$

**4.** [18 Points] Find the **sum** of each of the following series (which do converge). Simplify.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\ 3^{2n-1}}{4^{2n+1}} = -\frac{3}{4^3} + \frac{3^3}{4^5} - \frac{3^5}{4^7} + \ldots$

Here we have a nice geometric series with $a = -\dfrac{3}{64}$ and $r = -\dfrac{3^2}{4^2} = -\dfrac{9}{16}$

As a result, the sum is given by $\dfrac{a}{1-r} = \dfrac{-\dfrac{3}{64}}{1 - \left(-\dfrac{9}{16}\right)} = \dfrac{-\dfrac{3}{64}}{\dfrac{25}{16}} = -\dfrac{3}{64} \cdot \dfrac{16}{25} = \boxed{-\dfrac{3}{100}}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}\ (\ln 9)^n}{2^n\ n!} = -\sum_{n=0}^{\infty} \frac{\left(-\dfrac{\ln 9}{2}\right)^n}{n!} = -e^{-\frac{\ln 9}{2}} = e^{\ln\left(9^{-\frac{1}{2}}\right)} = -\frac{1}{\sqrt{9}} = \boxed{-\frac{1}{3}}$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{9^n\ (2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{3^{2n}\ (2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\dfrac{\pi}{3}\right)^{2n}}{(2n+1)!}$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \left(\dfrac{\pi}{3}\right)^{2n}}{(2n+1)!} \cdot \frac{\left(\dfrac{\pi}{3}\right)}{\left(\dfrac{\pi}{3}\right)} = \frac{3}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\dfrac{\pi}{3}\right)^{2n+1}}{(2n+1)!}$$

$$= \frac{3}{\pi} \sin\left(\frac{\pi}{3}\right) = \frac{3}{\pi}\left(\frac{\sqrt{3}}{2}\right) = \boxed{\frac{3\sqrt{3}}{2\pi}}$$

(d) $-1 + \dfrac{1}{3} - \dfrac{1}{5} + \dfrac{1}{7} - \dfrac{1}{9} + \ldots = -\left(1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \dfrac{1}{9} - \ldots\right) = -\arctan 1 = \boxed{-\dfrac{\pi}{4}}$

(e) $-\dfrac{\pi^2}{2!} + \dfrac{\pi^4}{4!} - \dfrac{\pi^6}{6!} + \dfrac{\pi^8}{8!} - \ldots = \cos\pi - 1 = -1 - 1 = \boxed{-2}$

(f) $\dfrac{1}{6} - \dfrac{1}{2(6)^2} + \dfrac{1}{3(6)^3} - \dfrac{1}{4(6)^4} + \ldots = \ln\left(1 + \dfrac{1}{6}\right) = \boxed{\ln\left(\dfrac{7}{6}\right)}$

**5.** [32 Points] In each case determine whether the given series is **absolutely convergent**, **conditionally convergent**, or **divergent**. Justify your answers.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\,(3+n^2)}{n^7+4}$

First, we show the absolute series $\displaystyle\sum_{n=1}^{\infty} \frac{3+n^2}{n^7+4}$ is convergent using LCT.

We see that $\displaystyle\sum_{n=1}^{\infty} \frac{3+n^2}{n^7+4} \approx \sum_{n=1}^{\infty} \frac{n^2}{n^7} = \sum_{n=1}^{\infty} \frac{1}{n^5}$ which is a convergent $p$-series $p = 5 > 1$.

Check: $\displaystyle\lim_{n\to\infty} \frac{\frac{3+n^2}{n^7+4}}{\frac{1}{n^5}} = \lim_{n\to\infty} \frac{3n^5+n^7}{n^7+4} = \lim_{n\to\infty} \frac{\frac{3}{n^2}+1}{1+\frac{4}{n^7}} = 1$ which is finite and non-zero. Therefore, these two series share the same behavior. Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^5}$ converges, then the absolute series also converges by LCT. Finally, we have $\boxed{\text{Absolute Convergence}}$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{6}{n^6} + \frac{\sin^2 n}{6^n} = \sum_{n=1}^{\infty} \frac{6}{n^6} + \sum_{n=1}^{\infty} \frac{\sin^2 n}{6^n}$

The first sum $\displaystyle\sum_{n=1}^{\infty} \frac{6}{n^6} = 6\sum_{n=1}^{\infty} \frac{1}{n^6}$ is convergent as a constant multiple of a convergent $p$-series with $p = 6 > 1$.

The second sum $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n}{6^n}$ is also convergent by CT.

The terms are bounded $\displaystyle\frac{\sin^2 n}{6^n} < \frac{1}{6^n}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{6^n}$ is a convergent geometric series

with $|r| = \left|\frac{1}{6}\right| = \frac{1}{6} < 1$.

Finally, sum of two convergent series is convergent. Here the O.S.=A.S. so we have $\boxed{\text{Absolute Convergence}}$ or $\boxed{\text{AC}}$.

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{n^3}{\ln n}$

$\boxed{\text{Diverges}}$ by the $n^{th}$ term Divergence Test since

$$\lim_{n\to\infty} \frac{n^3}{\ln n} = \lim_{x\to\infty} \frac{x^3}{\ln x} \left(\frac{\infty}{\infty}\right) \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{3x^2}{\left(\frac{1}{x}\right)} = \lim_{x\to\infty} 3x^3 = \infty \neq 0$$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\ n}{n^2+1}$

First, we show the absolute series is divergent. Note that $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1} \approx \sum_{n=1}^{\infty} \frac{1}{n}$ which is a divergent (Harmonic) $p$-series with $p=1$. Next,

Check: $\displaystyle\lim_{n\to\infty} \frac{\frac{n}{n^2+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^2}} = 1$ which is finite and non-zero.

Therefore the A.S. is also divergent by LCT.

Secondly, we are left to examine the original alternating series with the Alternating Series Test.

- $b_n = \dfrac{n}{n^2+1} > 0$

- $\displaystyle\lim_{n\to\infty} \frac{n}{n^2+1} = 0$

- $\dfrac{1}{b_{n+1}} < \dfrac{1}{b_n}$

because the related function $f(x) = \dfrac{x}{x^2+1}$ has negative derivative $f'(x) = \dfrac{1-x^2}{(x^2+1)^2} < 0$ when $x > 2$.

Then the O.S. converges by the Alternating Series Test. Finally, O.S. is $\boxed{\text{Conditionally Convergent}}$ or $\boxed{\text{CC}}$.

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (3n)!\ \ln n}{(n!)^2\ 2^{4n}\ n^n}$

Try Ratio Test:

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\dfrac{(-1)^{n+1} \ln(n+1)\ (3(n+1))!}{(n+1)^{n+1} 2^{4(n+1)} ((n+1)!)^2}}{\dfrac{(-1)^n\ \ln n\ (3n)!}{n^n\ 2^{4n} (n!)^2}} \right|$$

$$= \lim_{n\to\infty} \left(\frac{(3n+3)!}{(3n)!}\right) \left(\frac{n^n}{(n+1)^{n+1}}\right) \left(\frac{(n!)^2}{((n+1)!)^2}\right) \left(\frac{2^{4n}}{2^{4n+4}}\right) \left(\frac{\ln(n+1)}{\ln n}\right)$$

$$= \lim_{n\to\infty} \left(\frac{(3n+3)(3n+2)(3n+1)(3n)!}{(3n)!}\right) \left(\frac{n^n}{(n+1)^n (n+1)}\right) \left(\frac{(n!)^2}{(n+1)^2 (n!)^2}\right) \left(\frac{2^{4n}}{2^{4n} 2^4}\right) \left(\frac{\ln(n+1)}{\ln n}\right)$$

$$\overset{(*)}{=} \lim_{n\to\infty} \left(\frac{3(n+1)(3n+2)(3n+1)}{1}\right) \left(\frac{1}{e}\right) \left(\frac{1}{n+1}\right) \left(\frac{1}{(n+1)^2}\right) \left(\frac{1}{16}\right) (1)$$

$$= \lim_{n\to\infty} \left(\frac{3}{16e}\right) \left(\frac{3n+2}{n+1}\right) \left(\frac{3n+1}{n+1}\right) = \lim_{n\to\infty} \left(\frac{3}{16e}\right) \left(\frac{3+\frac{2}{n}}{1+\frac{1}{n}}\right) \left(\frac{3+\frac{1}{n}}{1+\frac{1}{n}}\right)$$

$$= \frac{27}{16e} < 1$$

Therefore the original series $\boxed{\text{Converges Absolutely by the Ratio test}}$.

Here, from above,

$$(*) = \lim_{n\to\infty} \frac{\ln(n+1)}{\ln n} \overset{\frac{\infty}{\infty}}{} = \lim_{x\to\infty} \frac{\ln(x+1)}{\ln x} \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{\frac{1}{x+1}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{x}{x+1} \overset{\frac{\infty}{\infty}}{} \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{1}{1} = 1$$

**6.** [16 Points] Find the **Interval** and **Radius** of Convergence for the following power series. Analyze carefully and with full justification.

$$\sum_{n=1}^{\infty} \frac{(-1)^n \ (5x+1)^n}{n^9 \cdot 9^n}$$

Use Ratio Test.

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}(5x+1)^{n+1}}{(n+1)^9 9^{n+1}}}{\frac{(-1)^n(5x+1)^n}{n^9 9^n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(5x+1)^{n+1}}{(5x+1)^n} \right| \cdot \left( \frac{n}{n+1} \right)^9 \cdot \frac{9^n}{9^{n+1}} = \frac{|5x+1|}{9}$$

The Ratio Test gives convergence for $x$ when $\frac{|5x+1|}{9} < 1$ or $|5x+1| < 9$.

That is $-9 < 5x+1 < 9 \Longrightarrow -10 < 5x < 8 \Longrightarrow -2 < x < \frac{8}{5}$

Endpoints:

$\bullet x = -2$ The original series becomes $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \ (5\,(-2)+1))^n}{n^9 \ 9^n} = \sum_{n=0}^{\infty} \frac{(-1)^n \ (-9)^n}{n^9 \ 9^n}$

$= \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \ (-1)^n 9^n}{n^9 \ 9^n} = \sum_{n=0}^{\infty} \frac{1}{n^9}$ which is a convergent $p$-series with $p = 9 > 1$.

$\bullet x = \frac{8}{5}$ The original series becomes $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \ \left(5\left(\frac{8}{5}\right)+1\right))^n}{n^9 \ 9^n} = \sum_{n=0}^{\infty} \frac{(-1)^n \ 9^n}{n^9 \ 9^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n^9}$

which is **convergent** by the Absolute Convergence Test (ACT) since the absolute series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^9}$ is convergent, as shown above with the left endpoint.

Finally, Interval of Convergence $\boxed{I = \left[-2, \frac{8}{5}\right]}$ with Radius of Convergence $\boxed{R = \frac{9}{5}}$.

**7.** [10 Points] Use MacLaurin series to **Estimate** $\int_0^1 x^2 \arctan\left(x^2\right)\, dx$ with error less than $\frac{1}{50}$.

Please analyze with detail and justify carefully. Simplify.

$$\int_0^1 x^2 \arctan\left(x^2\right)\, dx = \int_0^1 x^2 \sum_{n=0}^{\infty} \frac{(-1)^n \left(x^2\right)^{2n+1}}{2n+1}\, dx = \int_0^1 x^2 \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{4n+2}}{2n+1}\, dx$$

$$= \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{4n+4}}{2n+1}\, dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+5}}{(2n+1)(4n+5)}\Bigg|_0^1$$

$$= \frac{x^5}{1\cdot 5} - \frac{x^9}{3\cdot 9} + \frac{x^{13}}{5\cdot 13} - \ldots \Bigg|_0^1 = \frac{x^5}{5} - \frac{x^9}{27} + \frac{x^{13}}{65} - \ldots \Bigg|_0^1$$

$$= \frac{1}{5} - \frac{1}{27} + \frac{1}{65} - \ldots - (0 - 0 + 0 - \ldots)$$

$$\approx \frac{1}{5} - \frac{1}{27} = \frac{27}{135} - \frac{5}{135} = \boxed{\frac{22}{135}} \leftarrow \text{Estimate}$$

Note this is an alternating series. Use the Alternating Series Estimation Theorem. If we approximate the actual sum with only the first two terms, the error from the actual sum will be at most the absolute value of the next (first neglected) term, $\frac{1}{65}$. Here the maximum error $\frac{1}{65} < \frac{1}{50}$ as desired.

**8.** [16 Points]

(a) Consider the region bounded by $y = 1 + \arctan x$, $y = \ln x$, $x = 1$ and $x = 2$. Rotate the region about the vertical line $\boxed{x = -2}$. Set-up, **BUT DO NOT EVALUATE!!**, the integral to compute the volume of the resulting solid using the Cylindrical Shells Method. Sketch the solid, along with one of the approximating shells.

See me for a sketch.

$$V = 2\pi \int_1^2 \text{radius height}\; dx = 2\pi \int_1^2 (x+2)\left(1 + \arctan x - \ln x\right)\, dx$$

(b) Consider the region bounded by $y = \arcsin x$, $y = \frac{\pi}{2}$, $x = 0$ and $x = 1$. Rotate the region about the vertical line $\boxed{x = 4}$. Set-up, **BUT DO NOT EVALUATE!!**, the integral to compute the volume of the resulting solid using the Cylindrical Shells Method. Sketch the solid, along with one of the approximating shells.

See me for a sketch.

$$V = 2\pi \int_0^1 \text{radius height}\; dx = 2\pi \int_0^1 (4-x)\left(\frac{\pi}{2} - \arcsin x\right)\, dx$$

**9.** [20 Points]

(a) Consider the Parametric Curve represented by $x = \frac{t^3}{3} - \frac{e^{2t}}{2}$ and $y = 2te^t - 2e^t$.

**COMPUTE** the **arclength** of this parametric curve for $0 \le t \le 1$. Simplify.

First, $\dfrac{dx}{dt} = t^2 - e^{2t}$ and $\dfrac{dy}{dt} = 2te^t + 2e^t - 2e^t = 2te^t$.

$$L = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt = \int_0^1 \sqrt{(t^2 - e^{2t})^2 + (2te^t)^2}\ dt$$

$$= \int_0^1 \sqrt{t^4 - 2t^2e^{2t} + e^{4t} + 4t^2e^{2t}}\ dt = \int_0^1 \sqrt{t^4 + 2t^2e^{2t} + e^{4t}}\ dt = \int_0^1 \sqrt{(t^2 + e^{2t})^2}\ dt$$

$$= \int_0^1 t^2 + e^{2t}\ dt = \frac{t^3}{3} + \frac{e^{2t}}{2}\bigg|_0^1 = \frac{1}{3} + \frac{e^2}{2} - \left(0 + \frac{e^0}{2}\right) = \frac{1}{3} + \frac{e^2}{2} - \frac{1}{2} = \frac{2}{6} + \frac{e^2}{2} - \frac{3}{6} = \boxed{\frac{e^2}{2} - \frac{1}{6}}$$

(b) Consider a different Parametric Curve represented by $x = \sin^3 t$ and $y = \cos^3 t$.

**COMPUTE** the **surface area** obtained by rotating this curve about the $y$-axis for $0 \le t \le \dfrac{\pi}{2}$. Simplify.

First, $\dfrac{dx}{dt} = 3\sin^2 t \cos t$ and $\dfrac{dy}{dt} = 3\cos^2 t(-\sin t) = -3\cos^2 t \sin t$

$$\text{S.A.} = \int_0^{\frac{\pi}{2}} 2\pi x(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$= \int_0^{\frac{\pi}{2}} 2\pi \sin^3 t\sqrt{\left(3\sin^2 t\cos t\right)^2 + (-3\cos^2 t\sin t)^2}\ dt$$

$$= 2\pi\int_0^{\frac{\pi}{2}} \sin^3 t\sqrt{9\sin^4 t\cos^2 t + 9\cos^4 t\sin^2 t}\ dt$$

$$= 2\pi\int_0^{\frac{\pi}{2}} \sin^3 t\sqrt{9\sin^2 t\cos^2 t(\sin^2 t + \cos^2 t)}\ dt = 2\pi\int_0^{\frac{\pi}{2}} \sin^3 t\sqrt{9\sin^2 t\cos^2 t(1)}\ dt$$

$$= 2\pi\int_0^{\frac{\pi}{2}} \sin^3 t(3\sin t\cos t)\ dt = 6\pi\int_0^{\frac{\pi}{2}} \sin^4 t\cos t\ dt = 6\pi\left(\frac{\sin^5 t}{5}\right)\bigg|_0^{\frac{\pi}{2}}$$

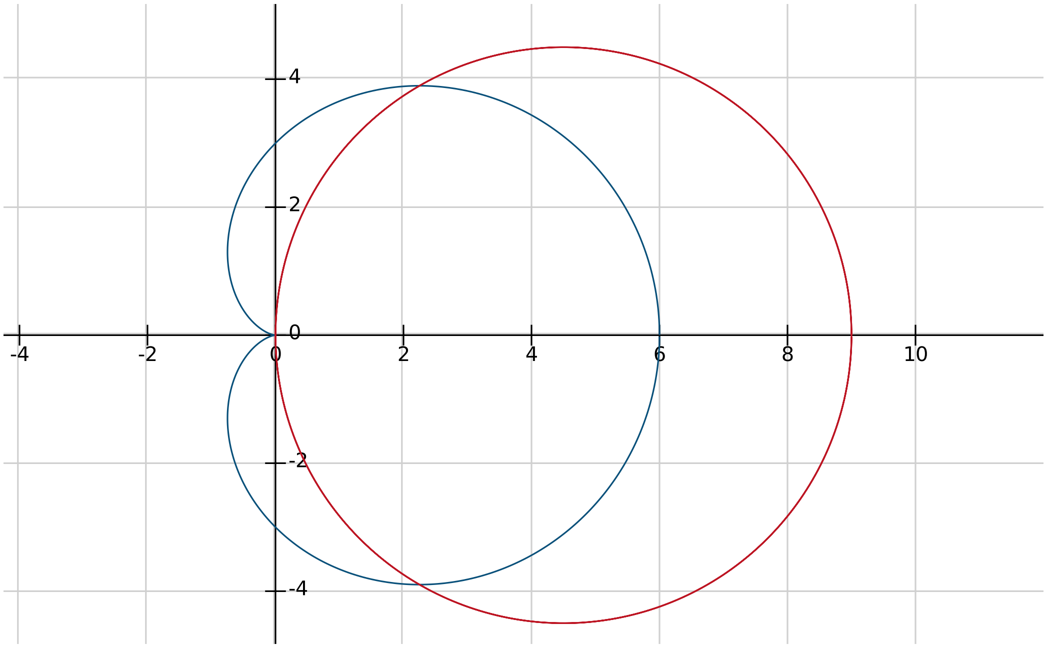
$$= 6\pi\left(\frac{\sin^5\left(\frac{\pi}{2}\right)}{5}\right) - \left(\frac{\sin^5 0}{5}\right) = 6\pi\left(\frac{1}{5} - 0\right) = \boxed{\frac{6\pi}{5}}$$

**10.** [20 Points] For each of the following parts, do the following **two** things:

1. Sketch the Polar curves and shade the described bounded region.

2. Set-Up but **DO NOT EVALUATE** the Integral representing the area of the described bounded region.

(a) The **area** bounded outside the polar curve $r = 3 + 3\cos\theta$ and inside the polar curve $r = 9\cos\theta$.




These two polar curves intersect when $3+3\cos\theta = 9\cos\theta \Rightarrow 6\cos\theta = 3 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \pm\dfrac{\pi}{3}$.

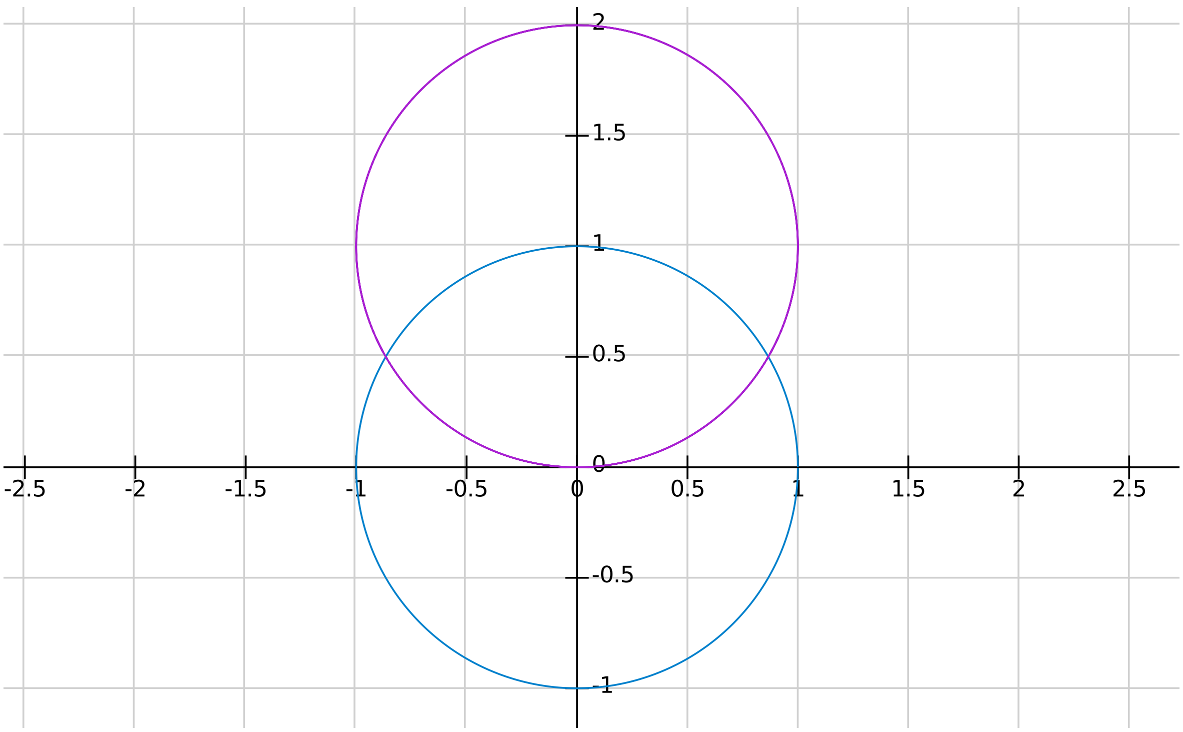
Area $= A = \dfrac{1}{2}\displaystyle\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left((\text{outer}\ \ r)^2 - (\text{inner}\ \ r)^2\right)\ d\theta$

$$= \boxed{\frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left((9\cos\theta)^2 - (3+3\cos\theta)^2\right)\ d\theta}$$

OR using symmetry

$$A = \boxed{2\left(\frac{1}{2}\int_{0}^{\frac{\pi}{3}} (9\cos\theta)^2 - (3+3\cos\theta)^2\ d\theta\right)}$$

(b) The **area** bounded outside the polar curve $r = 1$ and inside the polar curve $r = 2\sin\theta$.




These two polar curves intersect when

$2\sin\theta = 1 \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \dfrac{\pi}{6}$ or $\theta = \dfrac{5\pi}{6}$ .

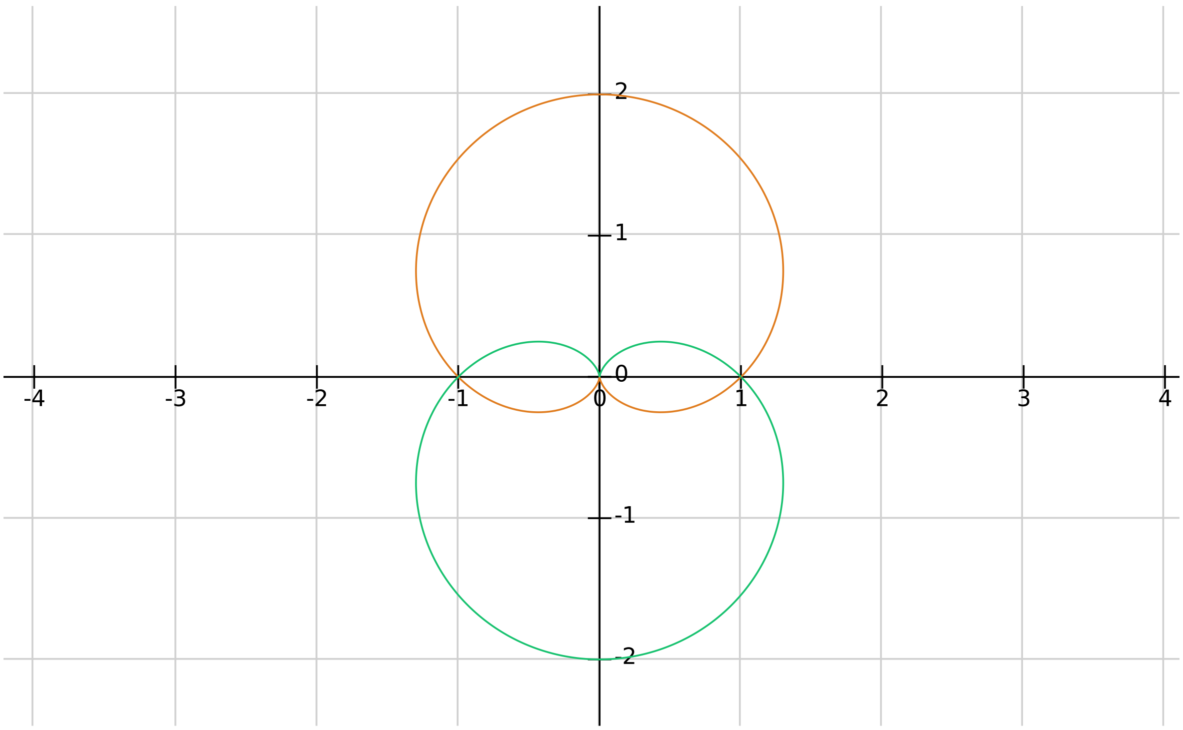
$$\text{Area} = A = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left((\text{outer}\ \ r)^2 - (\text{inner}\ \ r)^2\right)\ d\theta$$

$$= \boxed{\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (2\sin\theta)^2 - (1)^2\ d\theta}$$

OR using symmetry

$$A = 2\left(\frac{1}{2}\int_0^{\frac{\pi}{2}} (2\sin\theta)^2 - (1)^2 \; d\theta\right)$$

(c) The **area** that lies inside both of the curves $r = 1 + \sin\theta$ and inside the polar curve $r = 1 - \sin\theta$.




These two polar curves intersect when $\theta = 0$ and $\theta = \pi$.

Using symmetry, we have

$$\text{Area} = A = 4\left(\frac{1}{2}\int_0^{\frac{\pi}{2}} \left((\text{outer} \;\; r)^2 - (\text{inner} \;\; r)^2\right) \; d\theta\right)$$

$$= \boxed{4\left(\frac{1}{2}\int_0^{\frac{\pi}{2}} (1 - \sin\theta)^2 \; d\theta\right)}$$

OR you could use symmetry again

$$A = \boxed{4\left(\frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} (1 - \sin\theta)^2 \; d\theta\right)}$$

OR you could use symmetry again

$$A = \boxed{2\left(\frac{1}{2}\int_{0}^{\pi}(1-\sin\theta)^2\ d\theta\right)}$$

OR you could use symmetry again

$$A = \boxed{4\left(\frac{1}{2}\int_{\pi}^{\frac{3\pi}{2}}(1+\sin\theta)^2\ d\theta\right)}$$

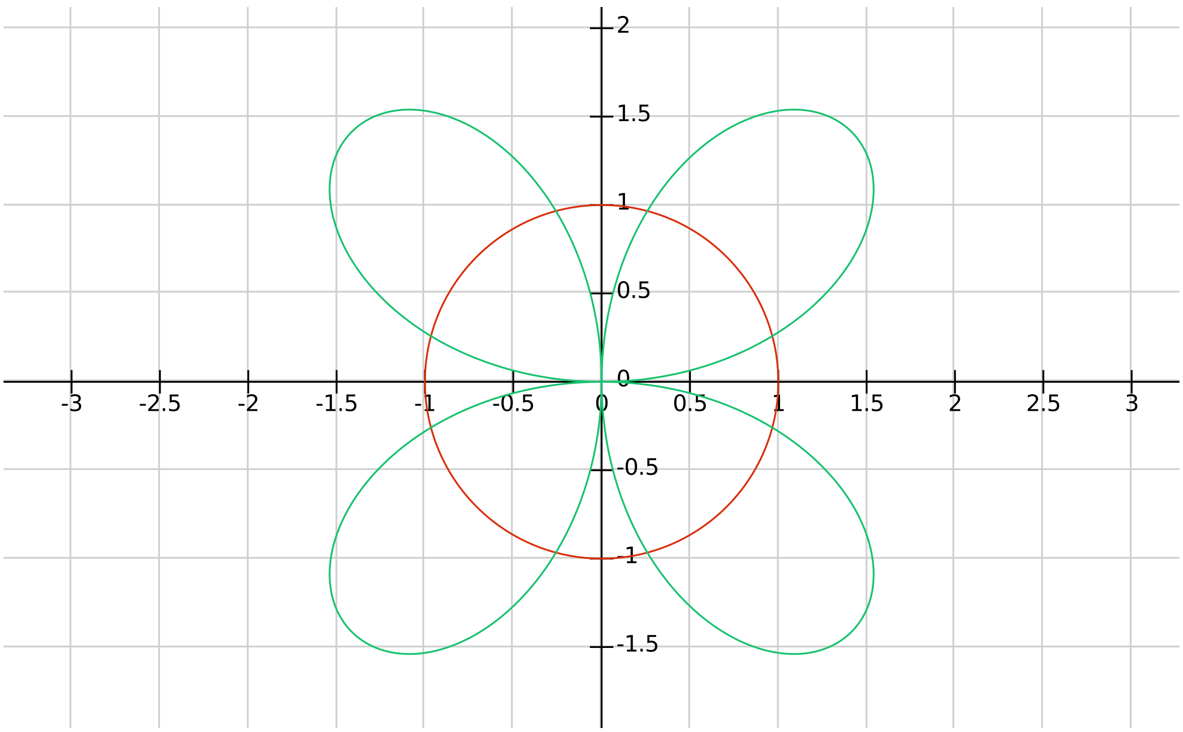
OR you could use symmetry again

$$A = \boxed{4\left(\frac{1}{2}\int_{\frac{3\pi}{2}}^{2\pi}(1+\sin\theta)^2\ d\theta\right)}$$

OR you could use symmetry again

$$A = \boxed{2\left(\frac{1}{2}\int_{\pi}^{2\pi}(1+\sin\theta)^2\ d\theta\right)}$$

(d) The **area** bounded outside the polar curve $r = 1$ and inside the polar curve $r = 2\sin(2\theta)$.




These two polar curves intersect when $2\sin(2\theta) = 1 \Rightarrow \sin(2\theta) = \frac{1}{2} \Rightarrow \theta = \dfrac{\pi}{12}$ or $\dfrac{5\pi}{12}$.

Using symmetry

$$\text{Area} = A = 4\left(\frac{1}{2}\int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} \left((\text{outer}\ \ r)^2 - (\text{inner}\ \ r)^2\right)\ d\theta\right)$$

$$= \boxed{4\left(\frac{1}{2}\int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} (2\sin(2\theta)^2 - (1)^2\ d\theta\right)}$$

OR using more symmetry

$$A = 8\left(\frac{1}{2}\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} (2\sin(2\theta)^2 - (1)^2\ d\theta\right)$$