Math 121 Final Exam Answer Key May 13, 2015

1. [15 Points] Evaluate each of the following **limits**. Please justify your answers. Be clear if the limit equals a value, $+\infty$ or $-\infty$, or Does Not Exist.

(a) $$\lim_{x\to 0} \frac{\ln(1-x)+x}{\cosh(4x)-\arctan(3x)-e^{-3x}} \overset{(\frac{0}{0})}{\underset{\text{L'H}}{=}} \lim_{x\to 0} \frac{-\frac{1}{1-x}+1}{4\sinh(4x)-\frac{3}{1+9x^2}+3e^{-3x}} \; (\tfrac{0}{0})$$

$$\overset{\text{L'H}}{=} \lim_{x\to 0} \frac{-\frac{1}{(1-x)^2}}{16\cosh(4x)+\frac{3(18x)}{(1+9x^2)^2}-9e^{-3x}} = -\frac{1}{16-9} = \boxed{-\frac{1}{7}}$$

(b) $$\lim_{x\to\infty} \left(e^{\frac{1}{x^3}} - \frac{5}{x^3}\right)^{x^3} {}^{1^\infty}$$

$$= \lim_{x\to\infty} e^{\ln\left(\left(e^{\frac{1}{x^3}} - \frac{5}{x^3}\right)^{x^3}\right)} = e^{\lim_{x\to\infty} \ln\left(\left(e^{\frac{1}{x^3}} - \frac{5}{x^3}\right)^{x^3}\right)}$$

$$= e^{\lim_{x\to\infty} x^3 \ln\left(\left(e^{\frac{1}{x^3}} - \frac{5}{x^3}\right)\right)}$$

$$= e^{\lim_{x\to\infty} \frac{\ln\left(e^{\frac{1}{x^3}} - \frac{5}{x^3}\right)}{\frac{1}{x^3}}} \; \tfrac{0}{0}$$

$$\overset{\text{L'H}}{=} e^{\lim_{x\to\infty} \frac{\frac{1}{e^{\frac{1}{x^3}} - \frac{5}{x^3}}\left(e^{\frac{1}{x^3}}\left(-\frac{3}{x^4}\right)+\frac{15}{x^4}\right)}{-\frac{3}{x^4}}}$$

$$= e^{\lim_{x\to\infty} \frac{1}{e^{\frac{1}{x^3}} - \frac{5}{x^3}}\left(e^{\frac{1}{x^3}} - 5\right)} = e^{1-5} = \boxed{e^{-4}}$$

2. [30 Points] Evaluate the following **integrals**.

(a) $$\int \frac{x^4+3x^3+6x^2+6x+5}{x^3+x^2+2x+2}\,dx = \int \frac{x^4+3x^3+6x^2+6x+5}{(x+1)(x^2+2)}\,dx$$

$$= \int x+2+\frac{2x^2+1}{(x+1)(x^2+2)}\,dx$$

$$= \int x+2+\frac{1}{x+1}+\frac{x-1}{x^2+2}\,dx = \int x+2+\frac{1}{x+1}+\frac{x}{x^2+2}-\frac{1}{x^2+2}\,dx$$

$$= \boxed{\frac{x^2}{2} + +2x + \ln|x+1| + \frac{\ln|x^2+2|}{2} - \frac{1}{\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right) + C}$$

Long division yields:

$$x^3+x^2+2x+2\overline{)x^4+3x^3+6x^+6x+5}^{\,x+2}$$

$$\underline{-(x^4+x^3+2x^2+2x)}$$

$$2x^3+4x^2+4x+5$$

$$\underline{-(2x^3+2x^2+4x+4)}$$

$$2x^2+1$$

Partial Fractions Decomposition:

$$\frac{2x^2+1}{(x+1)(x^2+2)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+2}$$

Clearing the denominator yields:

$2x^2+1 = A(x^2+2)+(Bx+C)(x+1)$
$2x^2+1 = Ax^2+2A+Bx^2+Bx+Cx+C$
$2x^2+1 = (A+B)x^2+(B+C)x+2A+C$
so that $A+B=2,\ B+C=0$ and $2A+C=1$
Solve for $A=1,\ B=1$ and $C=-1$

(b) $$\int_2^{2\sqrt{3}} \frac{1}{\sqrt{16-x^2}}\,dx = \arcsin\left(\frac{x}{4}\right)\Big|_2^{2\sqrt{3}} = \arcsin\left(\frac{2\sqrt{3}}{4}\right) - \arcsin\left(\frac{2}{4}\right)$$

$$= \arcsin\left(\frac{\sqrt{3}}{2}\right) - \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{3} - \frac{\pi}{6} = \boxed{\frac{\pi}{6}}$$

(c) $$\int \frac{x^2}{\sqrt{16-x^2}}\,dx = \int \frac{16\sin^2\theta}{\sqrt{16-16\sin^2\theta}}\,4\cos\theta\,d\theta$$

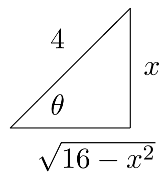
$$= \int \frac{16\sin^2\theta}{\sqrt{16(1-\sin^2\theta)}}\,4\cos\theta\,d\theta = \int \frac{16\sin^2\theta}{\sqrt{16\cos^2\theta}}\,4\cos\theta\,d\theta = \int \frac{16\sin^2\theta}{4\cos\theta}\,4\cos\theta\,d\theta$$

$$= 16\int \sin^2\theta\,d\theta = 16\int \frac{1-\cos(2\theta)}{2}\,d\theta = 8\int 1-\cos(2\theta)\,d\theta$$

$$= 8\left(\theta - \frac{\sin(2\theta)}{2}\right) + C = 8\left(\theta - \frac{2\sin\theta\cos\theta}{2}\right) + C = 8\left(\theta - \sin\theta\cos\theta\right) + C$$

$$= 8\left(\arcsin\left(\frac{x}{4}\right) - \left(\frac{x}{4}\right)\left(\frac{\sqrt{16-x^2}}{4}\right)\right) + C = \boxed{8\left(\arcsin\left(\frac{x}{4}\right) - \left(\frac{x\sqrt{16-x^2}}{16}\right)\right) + C}$$

Trig. Substitute $\boxed{\begin{array}{l} x = 4\sin\theta \\ dx = 4\cos\theta d\theta \end{array}}$



(d) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos x}{\left[1+\sin^2 x\right]^{\frac{7}{2}}}\ dx = \int_0^1 \frac{1}{[1+u^2]^{\frac{7}{2}}}\ du$

$\displaystyle = \int_{u=0}^{u=1} \frac{1}{[1+\tan^2\theta]^{\frac{7}{2}}}\ \sec^2\theta\ d\theta = \int_{u=0}^{u=1} \frac{1}{[\sec^2\theta]^{\frac{7}{2}}}\ \sec^2\theta\ d\theta$

$\displaystyle = \int_{u=0}^{u=1} \frac{1}{\left(\sqrt{\sec^2\theta}\right)^7}\ \sec^2\theta\ d\theta = \int_{u=0}^{u=1} \frac{1}{\sec^7\theta}\ \sec^2\theta\ d\theta$

$\displaystyle = \int_{u=0}^{u=1} \frac{1}{\sec^5\theta}\ d\theta = \int_{u=0}^{u=1} \cos^5\theta\ d\theta = \int_{u=0}^{u=1} \cos^4\theta\cos\theta\ d\theta$

$\displaystyle = \int_{u=0}^{u=1} \left(\cos^2\theta\right)^2\cos\theta\ d\theta = \int_{u=0}^{u=1} \left(1-\sin^2\theta\right)^2\cos\theta\ d\theta$

$\displaystyle = \int_{u=0}^{u=1} (1-w^2)^2\ dw = \int_{u=0}^{u=1} 1-2w^2+w^4\ dw = w - \frac{2w^3}{3} + \frac{w^5}{5}\Bigg|_{u=0}^{u=1}$

$\displaystyle = \sin\theta - \frac{2\sin^3\theta}{3} + \frac{\sin^5\theta}{5}\Bigg|_{u=0}^{u=1} = \frac{u}{\sqrt{u^2+1}} - \frac{2}{3}\left(\frac{u}{\sqrt{u^2+1}}\right)^3 + \frac{1}{5}\left(\frac{u}{\sqrt{u^2+1}}\right)^5\Bigg|_{u=0}^{u=1}$

$\displaystyle = \frac{1}{\sqrt{2}} - \frac{2}{3}\left(\frac{1}{\sqrt{2}}\right)^3 + \frac{1}{5}\left(\frac{1}{\sqrt{2}}\right)^5 - (0-0+0)$

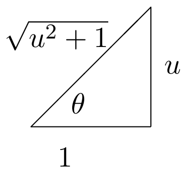
$\displaystyle = \frac{1}{\sqrt{2}} - \frac{2}{3}\left(\frac{1}{2\sqrt{2}}\right) + \frac{1}{5}\left(\frac{1}{4\sqrt{2}}\right) = \frac{1}{\sqrt{2}} - \frac{1}{3\sqrt{2}} + \frac{1}{20\sqrt{2}}$

$\displaystyle = \frac{60}{60\sqrt{2}} - \frac{20}{60\sqrt{2}} + \frac{3}{60\sqrt{2}} = \boxed{\frac{43}{60\sqrt{2}}}$

Standard u substitution to simplify at the start:

$\boxed{\begin{array}{l} u = \sin x \\ du = \cos x dx \end{array}}$ $\boxed{\begin{array}{l} x = 0 \Rightarrow u = 0 \\ x = \frac{\pi}{2} \Rightarrow u = 1 \end{array}}$

Trig. Substitute $\boxed{\begin{array}{l} u = \tan\theta \\ du = \sec^2\theta d\theta \end{array}}$



Standard w substitution for *odd* trig. integral $\displaystyle\int \cos^5\theta\ d\theta$ technique:

$$\boxed{\begin{array}{l} w = \sin\theta \\ \\ dw = \cos\theta\ d\theta \end{array}}$$

you can also change one more limit if you rather

$$\ldots = \int_0^{\frac{1}{\sqrt{2}}} (1-w^2)^2\ dw = \ldots$$

3. [25 Points] For each of the following **improper integrals**, determine whether it converges or diverges. If it converges, find its value.

(a) $\displaystyle\int_6^\infty \frac{1}{x^2-10x+28}\ dx = \lim_{t\to\infty}\int_6^t \frac{1}{x^2-10x+28}\ dx = \lim_{t\to\infty}\int_6^t \frac{1}{(x-5)^2+3}\ dx$

$$= \lim_{t\to\infty}\int_1^{t-5} \frac{1}{w^2+3}\ dx = \lim_{t\to\infty}\frac{1}{\sqrt{3}}\arctan\left(\frac{w}{\sqrt{3}}\right)\Bigg|_1^{t-5}$$

$$= \lim_{t\to\infty}\frac{1}{\sqrt{3}}\left(\arctan\left(\frac{t-5}{\sqrt{3}}\right) - \arctan\left(\frac{1}{\sqrt{3}}\right)\right)$$

$$= \frac{1}{\sqrt{3}}\left(\frac{\pi}{2}-\frac{\pi}{6}\right) = \boxed{\frac{\pi}{3\sqrt{3}}}$$

Substitute $\boxed{\begin{array}{l} w = x-5 \\ \\ dw = dx \end{array}}$ $\boxed{\begin{array}{l} x = 6 \Rightarrow w = 1 \\ \\ x = t \Rightarrow w = t-5 \end{array}}$

(b) $\displaystyle\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}\cdot\arcsin x}\ dx = \lim_{t\to 0^+}\int_t^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}\cdot\arcsin x}\ dx$

$$= \lim_{t\to 0^+}\int_{\arcsin t}^{\frac{\pi}{6}} \frac{1}{w}\ dw = \lim_{t\to 0^+}\ln|w|\,\Bigg|_{\arcsin t}^{\frac{\pi}{6}}$$

$$= \lim_{t\to 0^+}\ln\left|\frac{\pi}{6}\right| - \ln|\arcsin t| = \ln\left|\frac{\pi}{6}\right| - (-\infty) = \infty \qquad \text{Diverges}$$

$\boxed{\begin{array}{l} w = \arcsin x \\ \\ du = \dfrac{1}{\sqrt{1-x^2}}\ dx \end{array}}$ $\boxed{\begin{array}{l} x = t \Rightarrow w = \arcsin t \\ \\ x = \frac{1}{2} \Rightarrow w = \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6} \end{array}}$

(c) $\displaystyle\int_1^\infty \frac{1}{x^2+5x+6}\ dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2+5x+6}\ dx = \lim_{t\to\infty}\int_1^t \frac{1}{(x+2)(x+3)}\ dx$

$$\overset{\text{PFD}}{=} \lim_{t\to\infty}\int_1^t \frac{1}{x+2} - \frac{1}{x+3}\ dx = \lim_{t\to\infty}\ln|x+2| - \ln|x+3|\,\Bigg|_1^t = \lim_{t\to\infty}\ln|t+2| - \ln|t+3| - (\ln 3 - \ln 4)$$

$$= \lim_{t\to\infty} \ln\left|\frac{t+2}{t+3}\right| - (\ln 3 - \ln 4) = \lim_{t\to\infty} \ln\left|\frac{1+\frac{2}{t}}{1+\frac{3}{t}}\right| - (\ln 3 - \ln 4) = \boxed{\ln\left(\frac{4}{3}\right)}$$

note: or you can use L'H Rule to finish limit in the log.

Partial Fractions Decomposition:

$$\frac{1}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$$

Clearing the denominator yields:

$1 = A(x+3) + B(x+2)$
$1 = (A+B)x + (B+C)x + 3A + 2B$
so that $A + B = 0,$ and $3A + 2B = 1$
Solve for $A = 1,$ and $B = -1$

(d) $\displaystyle\int_0^1 x\ln x\, dx = \lim_{t\to 0^+}\int_t^1 x\ln x\, dx = \lim_{t\to 0^+} \frac{x^2}{2}\ln x\Big|_t^1 - \frac{1}{2}\int_t^1 x\, dx$

$\displaystyle= \lim_{t\to 0^+}\frac{x^2}{2}\ln x\Big|_t^1 - \frac{x^2}{4}\Big|_t^1 = \lim_{t\to 0^+}\frac{1}{2}\ln 1 - \frac{t^2}{2}\ln t - \left(\frac{1}{4} - \frac{t}{4}\right)$ see below (**) $= \boxed{-\frac{1}{4}}$ Converges

Integration By Parts:

$u = \ln x$	$dv = xdx$
$du = \frac{1}{x}dx$	$v = \frac{x^2}{2}$

(**) $\displaystyle\lim_{t\to 0^+} t^2\ln t \overset{0\cdot(-\infty)}{=} \lim_{t\to 0^+}\frac{\ln t}{\frac{1}{t^2}} \overset{\frac{-\infty}{\infty}}{=} \overset{L'H}{=} \lim_{t\to 0^+}\frac{\frac{1}{t}}{-\frac{2}{t^3}} = \lim_{t\to 0^+} -\frac{t^2}{2} = 0$

4. [15 Points] Find the **sum** of each of the following series (which do converge):

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n\ 7^{n+1}}{3^{3n-1}} = -\frac{7^2}{3^2} + \frac{7^3}{3^5} - \frac{7^4}{3^8} + \dots$

Here we have a geometric series with $a = -\frac{49}{9}$ and $r = -\frac{7}{3^3} = -\frac{7}{27}$

As a result, the sum is given by $\displaystyle\frac{a}{1-r} = \frac{-\frac{49}{9}}{1-\left(-\frac{7}{27}\right)} = \frac{-\frac{49}{9}}{\frac{34}{27}} = -\frac{49}{9}\cdot\frac{27}{34} = \boxed{-\frac{147}{34}}$

(b) $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\ 2^{n+1}\ (\ln 5)^n}{n!} = -2\sum_{n=0}^{\infty}\frac{(-1)^n\ 2^n\ (\ln 5)^n}{n!} = -2\sum_{n=0}^{\infty}\frac{(-2\ln 5)^n}{n!}$

$$= -2e^{-2\ln 5} = -2e^{\ln 5^{-2}} = \boxed{-\frac{2}{25}}$$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n-1}}{3\ (2n)!} = \frac{1}{3}\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n-1}}{(2n)!}\frac{\pi}{\pi} = \frac{1}{3\pi}\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{(2n)!} = \frac{1}{3\pi}\cos\pi = \boxed{-\frac{1}{3\pi}}$

(d) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots = \ln(1+1) = \boxed{\ln 2}$

(e) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots = \arctan(1) = \boxed{\frac{\pi}{4}}$

5. [35 Points] In each case determine whether the given series is **absolutely convergent**, **conditionally convergent**, or **divergent**. Justify your answers.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\ (n^3+7)}{n^7+3}$

First examine the absolute series $\displaystyle\sum_{n=1}^{\infty} \frac{n^3+7}{n^7+3} \approx \sum_{n=1}^{\infty} \frac{n^3}{n^7} = \sum_{n=1}^{\infty} \frac{1}{n^4}$

which is a convergent p-series with $p = 4 > 1$.

Next check

Check: $\displaystyle\lim_{n\to\infty} \frac{\frac{n^3+7}{n^7+3}}{\frac{1}{n^4}} = \lim_{n\to\infty} \frac{n^7+7n^4}{n^7+3} = \lim_{n\to\infty} \frac{1+\frac{7}{n^3}}{1+\frac{3}{n^7}} = 1$ which is finite and non-zero $(0 < 1 < \infty)$.

Therefore, these two series share the same behavior, and the absolute series $\displaystyle\sum_{n=1}^{\infty} \frac{n^3+7}{n^7+3}$ is also convergent by Limit Comparison Test (LCT). Then the Original Series is Convergent by ACT. Finally, we have $\boxed{\text{Absolute Convergence}}$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \arctan(7n)}{n^7+7}$

First we analyze the absolute series $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan(7n)}{n^7+7}$

We can bound the terms here:

$\displaystyle\frac{\arctan(7n)}{n^7+7} < \frac{\frac{\pi}{2}}{n^7+7} < \frac{\frac{\pi}{2}}{n^7}.$

Note that

$\displaystyle\sum_{n=1}^{\infty} \frac{\frac{\pi}{2}}{n^7} = \frac{\pi}{2}\sum_{n=1}^{\infty} \frac{1}{n^7}$ is a constant multiple of a convergent p-series with $p = 7 > 1$ and therefore, is convergent.

Therefore the absolute series converges by CT. The original series convergese by ACT. Finally, we have $\boxed{\text{Absolute Convergence}}$.

(c) $\displaystyle\sum_{n=1}^{\infty} n \cdot \arcsin\left(\frac{1}{n}\right)$

$\boxed{\text{Diverges}}$ by the n^{th} term Divergence Test since

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n \cdot \arcsin\left(\frac{1}{n}\right) \overset{\infty\cdot 0}{=} \lim_{x\to\infty} x \cdot \arcsin\left(\frac{1}{x}\right)$$

$$= \lim_{x\to\infty} \frac{\arcsin\left(\frac{1}{x}\right)}{\frac{1}{x}} \overset{\frac{0}{0}}{\underset{\text{L'H}}{=}} \lim_{x\to\infty} \frac{\frac{1}{\sqrt{1-\frac{1}{x^2}}}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{1}{\sqrt{1-\frac{1}{x^2}}} = 1 \neq 0$$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\, e^{3n}\, (2n)!}{n^n\, 4^{2n}\, (n!)^2}$

Try Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+2}e^{3n+3}(2(n+1))!}{(n+1)^{n+1}4^{2n+2}((n+1)!)^2}}{\frac{(-1)^{n+1}e^{3n}(2n)!}{n^n4^{2n}(n!)^2}}\right|$$

$$= \lim_{n\to\infty} \frac{e^{3n+3}}{e^{3n}} \cdot \frac{(2n+2)!}{(2n)!} \cdot \frac{n^n}{(n+1)^{n+1}} \cdot \frac{4^{2n}}{4^{2n+2}} \cdot \frac{(n!)^2}{((n+1)!)^2}$$

$$= \lim_{n\to\infty} \frac{e^{3n}e^3}{e^{3n}} \cdot \frac{(2n+2)(2n+1)(2n)!}{(2n)!} \cdot \frac{n^n}{(n+1)^n(n+1)} \cdot \frac{4^{2n}}{4^{2n}4^2} \cdot \frac{(n!)^2}{(n+1)^2(n!)^2}$$

$$= \lim_{n\to\infty} \frac{e^3}{1} \cdot \frac{2(n+1)(2n+1)}{1} \cdot \frac{n^n}{(n+1)^n} \cdot \frac{1}{n+1} \cdot \frac{1}{16} \cdot \frac{1}{(n+1)^2}$$

$$= \lim_{n\to\infty} \frac{e^3}{1} \cdot \frac{2(2n+1)}{1} \cdot \frac{1}{e} \cdot \frac{1}{16} \cdot \frac{1}{(n+1)^2}$$

$$= \lim_{n\to\infty} \frac{e^2}{8} \cdot \frac{2n+1}{(n+1)^2} = 0 < 1$$

Therefore, the series $\boxed{\text{Converges Absolutely}}$ by the Ratio Test.

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\, n}{n^2+4}$

First, we show the absolute series is divergent. Note that $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+4} \approx \sum_{n=1}^{\infty} \frac{1}{n}$ which is a divergent p-series with $p = 1$. Next, Check: $\lim_{n\to\infty} \dfrac{\frac{n}{n^2+4}}{\frac{1}{n}} = \lim_{n\to\infty} \dfrac{n^2}{n^2+4} \lim_{n\to\infty} \dfrac{1}{1+\frac{4}{n^2}} = 1$ which is finite and non-zero. Therefore, these two series share the same behavior. Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, then $\sum_{n=1}^{\infty} \frac{n}{n^2+4}$ is also divergent by Limit Comparison Test. As a result, we have no chance for Absolute Convergence.

Secondly, we are left to examine the original alternating series with the Alternating Series Test.

- $b_n = \dfrac{n}{n^2+4} > 0$

- $\lim_{n\to\infty} \dfrac{n}{n^2+4} = 0$

- $\dfrac{1}{b_{n+1}} < \dfrac{1}{b_n}$

because the related function $f(x) = \dfrac{x}{x^2+4}$ has negative derivative $f'(x) = \dfrac{4-x^2}{(x^2+4)^2} < 0$ when $x > 2$.

Therefore, the original series converges by the Alternating Series Test. Finally, we can conclude the original series is $\boxed{\text{Conditionally Convergent}}$

6. [15 Points] Find the **Interval** and **Radius** of Convergence for the following power series. Analyze carefully and with full justification.

$$\sum_{n=1}^{\infty} \frac{(-1)^n\ (\ln n)\ (4x-1)^n}{n^2 \cdot 5^n}$$

Use Ratio Test.

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\dfrac{(-1)^{n+1}\ln(n+1)(4x-1)^{n+1}}{(n+1)^2 5^{n+1}}}{\dfrac{(-1)^n \ln n(4x-1)^n}{(n+1)5^n}} \right|$$

$$= \lim_{n\to\infty} \frac{\ln(n+1)}{\ln n} \frac{|4x-1|}{5} \overset{(**)}{=} \frac{|4x-1|}{5}$$

The Ratio Test gives convergence for x when $\dfrac{|4x-1|}{5} < 1$ or $|4x-1| < 5$.

That is $-5 < 4x-1 < 5 \Longrightarrow -4 < 4x < 6 \Longrightarrow -1 < x < \dfrac{3}{2}$

$(**)$ $\lim_{n\to\infty} \dfrac{\ln(n+1)}{\ln n}^{\frac{\infty}{\infty}} = \lim_{x\to\infty} \dfrac{\ln(x+1)}{\ln x} \overset{\text{L'H}}{=} \lim_{x\to\infty} \dfrac{\frac{1}{x+1}}{\frac{1}{x}} = \lim_{x\to\infty} \dfrac{x}{x+1} = \lim_{x\to\infty} \dfrac{1}{1+\frac{1}{x}}$

Endpoints:

- $x = -1$ The original series becomes $\sum_{n=1}^{\infty} \dfrac{(-1)^n\ (\ln n)\ (-5)^n}{n^2 \cdot 5^n} = \sum_{n=1}^{\infty} \dfrac{(\ln n)5^n}{n^2 5^n} = \sum_{n=1}^{\infty} \dfrac{\ln n}{n^2}$

We bound $\dfrac{\ln n}{n^2} < \dfrac{\sqrt{n}}{n^2} = \dfrac{1}{n^{\frac{3}{2}}}$ and

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ is a convergent p-series with $p = \dfrac{3}{2} > 1$. Then our endpoint series converges by CT.

$\bullet x = \dfrac{3}{2}$ The original series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\ (\ln n)\ (5)^n}{n^2 \cdot 5^n} = \sum_{n=1}^{\infty} \frac{(-1)^n (\ln n)}{n^2}$

Here we consider the absolute series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$.

It is convergent as shown directly above. Therefore this alternating endpoint series is convergent by ACT.

Finally, Interval of Convergence $\boxed{I = \left[-1, \frac{3}{2}\right]}$ with Radius of Convergence $\boxed{R = \frac{5}{4}}$.

7. [8 Points]

(a) Write the MacLaurin Series for $f(x) = x^4 \arctan(2x)$. State the Radius of Convergence for this series.

First $\displaystyle\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$

Next $\displaystyle\arctan(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{2n+1}$

Finally $\displaystyle x^4 \arctan(2x) = x^4 \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+5}}{2n+1}$.

Here the Radius of convergence is $\boxed{\frac{1}{2}}$ because we need $|2x| < 1$.

(b) Use this series to determine the **seventh**, **eighth** and **ninth** derivatives of $f(x) = x^4 \arctan(2x)$ evaluated at $x = 0$. Do Not Simplify your answers here in part (b).

From above

$$x^4 \arctan(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+5}}{2n+1} = \frac{2x^5}{1} - \frac{2^3 x^7}{3} + \frac{2^5 x^9}{5} - \ldots$$

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^5(0)}{5!}x^5 + \frac{f^{(6)}(0)}{6!}x^6 + \frac{f^7(0)}{7!}x^7 + \frac{f^{(8)}(0)}{8!}x^8 + \frac{f^9(0)}{9!}x^9 + \ldots$$

Match coefficients of like degreed terms:

$$\frac{f^{(7)}(0)}{7!} = -\frac{8}{3} \Rightarrow \boxed{f^{(7)}(0) = -\frac{7!(8)}{3}} = -\frac{8!}{3}$$

$\dfrac{f^{(8)}(0)}{8!} = 0$ since the is no x^8 term $\Rightarrow \boxed{f^{(8)}(0) = 0}$

$\dfrac{f^{(9)}(0)}{9!} = \dfrac{32}{5} \Rightarrow \boxed{f^{(9)}(0) = \dfrac{9!(32)}{5}}$

8. [12 Points] Please analyze with detail and justify carefully. Simplify your answers.

(a) Estimate $e^{-\frac{1}{3}}$ with error less than $\dfrac{1}{100}$. Justify in words that your error is indeed less than $\dfrac{1}{100}$.

$e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$

So $e^{-\frac{1}{3}} = 1 + x + \dfrac{\left(-\frac{1}{3}\right)^2}{2!} + \dfrac{\left(-\frac{1}{3}\right)^3}{3!} + \dfrac{\left(-\frac{1}{3}\right)^4}{4!} + \dots = 1 - \dfrac{1}{3} + \dfrac{\frac{1}{9}}{2!} - \dfrac{\frac{1}{27}}{3!} + \dots$

$= 1 - \dfrac{1}{3} + \dfrac{1}{18} - \dfrac{1}{162} + \dots \approx 1 - \dfrac{1}{3} + \dfrac{1}{18} = \boxed{\dfrac{13}{18}} \leftarrow$ estimate

Using ASET we can estimate the full sum using only the first three terms with error *at most* $\dfrac{1}{162} < \dfrac{1}{100}$ as desired.

(b) Estimate $\arctan\left(\dfrac{1}{2}\right)$ with error less than $\dfrac{1}{100}$. Justify in words that your error is indeed less than $\dfrac{1}{100}$.

$\arctan x = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$

$\arctan\left(\dfrac{1}{2}\right) = \dfrac{1}{2} - \dfrac{\left(\frac{1}{2}\right)^3}{3} + \dfrac{\left(\frac{1}{2}\right)^5}{5} - \dfrac{\left(\frac{1}{2}\right)^7}{7} + \dots$

$= \dfrac{1}{2} - \dfrac{\frac{1}{8}}{3} + \dfrac{\frac{1}{32}}{5} - \dots = \dfrac{1}{2} - \dfrac{1}{24} + \dfrac{1}{160} - \dots \approx \dfrac{1}{2} - \dfrac{1}{24} = \dfrac{11}{24} \leftarrow$ estimate

Using ASET we can estimate the full sum using only the first two terms with error *at most* $\dfrac{1}{160} < \dfrac{1}{100}$ as desired.

(c) Estimate $\cos(1)$ with error less than $\dfrac{1}{10}$. Justify in words that your error is indeed less than $\dfrac{1}{10}$.

$\cos x = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

$\cos 1 = 1 - \dfrac{1}{2!} + \dfrac{1}{4!} - \dfrac{1}{6!} + \dots \approx 1 - \dfrac{1}{2} = \dfrac{1}{2} \leftarrow$ estimate

Using ASET we can estimate the full sum using only the first two terms with error *at most* $\dfrac{1}{24} < \dfrac{1}{10}$

as desired.

9. [15 Points]

(a) Consider the region bounded by $y = e^x - 1,\ \ y = 3,\ \ x = 0$. Rotate the region about the vertical line $\boxed{\boldsymbol{x = -1}}$. **Set-Up** but **DO NOT EVALUATE** the integral representing the **volume** of the resulting solid using the Cylindrical Shells Method. Sketch the solid, along with one of the approximating cylindrical shells.

See me for a full sketch.

Intersect?

$e^x - 1 = 3 \Rightarrow e^x = 4 \Rightarrow x = \ln 4$.

$$V = \int_0^{\ln 4} 2\pi \text{ radius height } dx = 2\pi \int_0^{\ln 4} (x+1)(3 - (e^x - 1))\ dx = \boxed{2\pi \int_0^{\ln 4} (x+1)(4 - e^x)\ dx}$$

(b) Consider the region bounded by $y = \arcsin x,\ \ y = 1$, and $x = 0$. Rotate the region about the vertical line $\boxed{\boldsymbol{x = 5}}$. **Set-Up** but **DO NOT EVALUATE** the integral representing the **volume** of the resulting solid using the Cylindrical Shells Method. Sketch the solid, along with one of the approximating cylindrical shells.

See me for a full sketch.

Intersect?

$\arcsin x = 1 \Rightarrow x = \sin 1$

$$V = \int_0^{\sin 1} 2\pi \text{ radius height } dx = \boxed{2\pi \int_0^{\sin 1} (5 - x)(1 - \arcsin x)\ dx}$$

(c) Consider the region bounded by $y = \arctan x,\ \ y = 4,\ \ x = 0$ and $x = 1$. Rotate the region about the $\boxed{y\text{-axis}}$. **COMPUTE** the **volume** of the resulting solid using the Cylindrical Shells Method. Sketch the solid, along with one of the approximating cylindrical shells.

See me for a full sketch.

$$V = \int_0^1 2\pi \text{ radius height } dx = 2\pi \int_0^1 x(4 - \arctan x)dx = 2\pi \int_0^1 4x - x \arctan x)dx$$

$$= 2\pi\left(2x^2 - \left(\frac{x^2}{2}\arctan x - \frac{x}{2} + \frac{1}{2}\arctan x\right)\right)\Bigg|_0^1 = 2\pi\left(2x^2 - \frac{x^2}{2}\arctan x + \frac{x}{2} - \frac{1}{2}\arctan x\right)\Bigg|_0^1$$

$$= 2\pi\left(2 - \frac{1}{2}\arctan 1 + \frac{1}{2} - \frac{1}{2}\arctan 1 - (0 - 0 + 0 - 0)\right) = 2\pi\left(2 - \frac{1}{2}\left(\frac{\pi}{4}\right) + \frac{1}{2} - \frac{1}{2}\left(\frac{\pi}{4}\right)\right)$$

$$= \boxed{2\pi\left(\frac{5}{2} - \frac{\pi}{4}\right)}$$

****

$$(**) \int x\ \arctan x\ dx = \frac{x^2}{2}\arctan x - \frac{1}{2}\int \frac{x^2 + 1 - 1}{1 + x^2}\ dx$$

$$= \frac{x^2}{2}\arctan x - \frac{1}{2}\int \frac{x^2+1}{1+x^2} - \frac{1}{1+x^2}\ dx = \frac{x^2}{2}\arctan x - \frac{1}{2}\int 1 - \frac{1}{1+x^2}\ dx$$

$$= \frac{x^2}{2}\arctan x - \frac{1}{2}(x - \arctan x) + C = \boxed{\frac{x^2}{2}\arctan x - \frac{1}{2}x + \frac{1}{2}\arctan x + C}$$

$$\boxed{\begin{array}{ll} u = \arctan x & dv = xdx \\ \\ du = \frac{1}{1+x^2}dx & v = \frac{x^2}{2} \end{array}}$$

OR if you don't like the "slip-in/slip out" technique, use a tangent trig. substitution instead to finish the second piece of the I.B.P. $\int \frac{x^2}{1+x^2}\ dx$

$$\int \frac{x^2}{1+x^2}\ dx = \int \frac{\tan^2\theta}{1+\tan^2\theta}\sec^2\theta\ d\theta = \int \frac{\tan^2\theta}{\sec^2\theta}\sec^2\theta\ d\theta = \int \tan^2\theta\ d\theta = \int \sec^2\theta - 1\ d\theta$$

$$= \tan\theta - \theta = x - \arctan x + C$$

Trig. Substitute $\boxed{\begin{array}{l} x = \tan\theta \\ \\ dx = \sec^2\theta d\theta \end{array}}$

10. [15 Points]

(a) Consider the Parametric Curve represented by $x = (\arctan t) - t$ and $y = 2\sinh^{-1} t$.

COMPUTE the **arclength** of this parametric curve for $0 \le t \le \sqrt{3}$.

$$\text{Recall } \frac{d}{dx}\sinh^{-1} x = \frac{1}{\sqrt{1+x^2}}$$

First $\frac{dx}{dt} = \frac{1}{1+t^2} - 1$ and $\frac{dy}{dt} = \frac{2}{\sqrt{1+t^2}}$.

$$L = \int_0^{\sqrt{3}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt = \int_0^{\sqrt{3}} \sqrt{\left(\frac{1}{1+t^2} - 1\right)^2 + \left(\frac{2}{\sqrt{1+t^2}}\right)^2}\ dt$$

$$= \int_0^{\sqrt{3}} \sqrt{\frac{1}{(1+t^2)^2} - \frac{2}{1+t^2} + 1 + \frac{4}{1+t^2}}\ dt = \int_0^{\sqrt{3}} \sqrt{\frac{1}{(1+t^2)^2} + \frac{2}{1+t^2} + 1}\ dt$$

$$= \int_0^{\sqrt{3}} \sqrt{\left(\frac{1}{1+t^2} + 1\right)^2}\ dt = \int_0^{\sqrt{3}} \frac{1}{1+t^2} + 1\ dt = \arctan t + t\Big|_0^{\sqrt{3}} = \arctan\sqrt{3} + \sqrt{3} - (0 - 0)$$

$$= \boxed{\frac{\pi}{3} + \sqrt{3}}$$

(b) Consider a *different* Parametric Curve represented by $x = t + \frac{1}{t}$ and $y = \ln\left(t^2\right)$.

COMPUTE the **surface area** obtained by rotating this curve about the $\boxed{y\text{-axis}}$, for $1 \le t \le 2$.

First $\dfrac{dx}{dt} = 1 - \dfrac{1}{t^2}$ and $\dfrac{dy}{dt} = \dfrac{2}{t}$.

$$\text{S.A.} = \int_1^2 2\pi x(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = 2\pi \int_1^2 \left(t + \frac{1}{t}\right)\sqrt{\left(1 - \frac{1}{t^2}\right)^2 + \left(\frac{2}{t}\right)^2}\, dt$$

$$= 2\pi \int_1^2 \left(t + \frac{1}{t}\right)\sqrt{1 - \frac{2}{t^2} + \frac{1}{t^4} + \frac{4}{t^2}}\, dt = 2\pi \int_1^2 \left(t + \frac{1}{t}\right)\sqrt{1 + \frac{2}{t^2} + \frac{1}{t^4}}\, dt$$

$$= 2\pi \int_1^2 \left(t + \frac{1}{t}\right)\sqrt{\left(1 + \frac{1}{t^2}\right)^2}\, dt = 2\pi \int_1^2 \left(t + \frac{1}{t}\right)\left(1 + \frac{1}{t^2}\right)\, dt$$

$$= 2\pi \int_1^2 t + \frac{2}{t} + \frac{1}{t^3}\, dt = 2\pi \left(\frac{t^2}{2} + 2\ln|t| - \frac{1}{2t^2}\right)\Bigg|_1^2$$

$$= 2\pi \left(2 + 2\ln 2 - \frac{1}{8} - \left(\frac{1}{2} + 2\ln 1 - \frac{1}{2}\right)\right) = 2\pi\left(2 + 2\ln 2 - \frac{1}{8}\right) = \boxed{2\pi\left(\frac{15}{8} + 2\ln 2\right)}$$

11. [15 Points] Compute the **area** bounded outside the polar curve $\boldsymbol{r = 1 + \sin\theta}$ and inside the polar curve $\boldsymbol{r = 3\sin\theta}$. **Sketch** the Polar curves **and** shade the bounded area.

These two polar curves intersect when

$1 + \sin\theta = 3\sin\theta \Rightarrow 2\sin\theta = 1 \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \dfrac{\pi}{6}$ or $\theta = \dfrac{5\pi}{6}$.

Using symmetry, we will integrate from $\theta = \dfrac{\pi}{6}$ to $\theta = \dfrac{\pi}{2}$ and double that area.

$$\text{Area} = A = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left((\text{outer } r)^2 - (\text{inner } r)^2\right)\, d\theta$$

$$= 2\left(\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left((\text{outer } r)^2 - (\text{inner } r)^2\right)\, d\theta\right)$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left((3\sin\theta)^2 - (1 + \sin\theta)^2\right)\, d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 9\sin^2\theta - (1 + 2\sin\theta + \sin^2\theta)\, d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 9\sin^2\theta - 1 - 2\sin\theta - \sin^2\theta\, d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 8\sin^2\theta - 1 - 2\sin\theta\, d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 8\left(\frac{1 - \cos(2\theta)}{2}\right) - 1 - 2\sin\theta\, d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 4\left(1 - \cos(2\theta)\right) - 1 - 2\sin\theta\, d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 4 - 4\cos(2\theta) - 1 - 2\sin\theta\, d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 3 - 4\cos(2\theta) - 2\sin\theta\, d\theta$$

$$= 3\theta - 2\sin(2\theta) + 2\cos\theta\Bigg|_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \left(3\left(\frac{\pi}{2}\right) - 2\sin\left(\frac{2\pi}{2}\right) + 2\cos\left(\frac{\pi}{2}\right)\right) - \left(3\left(\frac{\pi}{6}\right) - 2\sin\left(\frac{2\pi}{6}\right) + 2\cos\left(\frac{\pi}{6}\right)\right)$$

$$= \frac{3\pi}{2} - 2(0) + 2(0) - \left(\frac{\pi}{2} - 2\left(\frac{\sqrt{3}}{2}\right) + 2\left(\frac{\sqrt{3}}{2}\right)\right) = \boxed{\pi}$$