Name: Answer Key

Amherst College

DEPARTMENT OF MATHEMATICS

Math 121 Final Exam

May 9, 2017

- This is a closed-book examination. No books, notes, calculators, cell phones, communication devices of any sort, or other aids are permitted.

- You need *not* simplify algebraically complicated answers. However, numerical answers such as $\sin\left(\frac{\pi}{6}\right)$, $4^{\frac{3}{2}}$, $e^{\ln 4}$, $\ln(e^7)$, $e^{-\ln 5}$, $e^{3\ln 3}$, $\arctan(\sqrt{3})$, or $\cosh(\ln 3)$ should be simplified.

- Please *show* all of your work and *justify* all of your answers. (You may use the backs of pages for additional work space.)

Problem	Score	Possible Points
1		12
2		16
3		40
4		20
5		30
6		12
7		12
8		8
9		18
10		14
11		18
Total		200

$-5(1+25x^2)^{-1}$

1. [12 Points] Evaluate the following **limit**. Please justify your answer. Be clear if the limit equals a value, +∞ or −∞, or Does Not Exist. Simplify.

(a) $$\lim_{x\to 0}\frac{5xe^x-\arctan(5x)}{\sinh x+\ln(1-x)} \overset{0/0}{\underset{L'H}{=}} \lim_{x\to 0}\frac{5xe^x+5e^x-\frac{1}{1+25x^2}(5)}{\cosh x-\frac{1}{1-x}} \quad 0/0$$

$\rightsquigarrow -(1-x)^{-1}$

$$\underset{L'H}{=} \lim_{x\to 0}\frac{(\cancel{5xe^x}^{\,0}+5e^x)+5e^x+\cancel{\frac{5}{(1+25x^2)^2}(50x)}^{\,0}}{\cancel{\sinh x}^{\,0}-\frac{1}{(1-x)^2}}$$

$$=\frac{5+5}{-1}=\boxed{-10}$$

(b) Compute $$\lim_{x\to\infty}\left(1-\arcsin\left(\frac{5}{x}\right)\right)^x \overset{1^\infty}{=} e^{\lim\limits_{x\to\infty}\ln\left[\left(1-\arcsin\left(\frac{5}{x}\right)\right)^x\right]}$$

$$= e^{\lim\limits_{x\to\infty} x\ln\left(1-\arcsin\left(5/x\right)\right)} = e^{\lim\limits_{x\to\infty}\frac{\ln(1-\arcsin(5/x))}{1/x}} \quad 0/0$$

$\infty\cdot 0$ ⤳ Flip

$$\underset{L'H}{=} e^{\lim\limits_{x\to\infty}\frac{\cancel{\frac{1}{1-\arcsin(5/x)}}^{\,1}\cdot\left[\frac{(-)\,\cancel{1}^{\,1}}{\sqrt{1-(5/x)^2}}\right]\cancel{\left(-5/x^2\right)}}{\cancel{-1/x^2}}} = e^{-5} = \boxed{\frac{1}{e^5}}$$

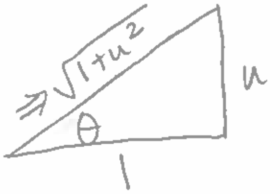
2. [16 Points] Evaluate the following integral.

(a) $$\int \frac{\cos x}{(1+\sin^2 x)^{\frac{7}{2}}}\,dx = \int \frac{1}{\left(\sqrt{1+u^2}\right)^7}\,du = \int \frac{1}{\left(\sqrt{1+\tan^2\theta}\right)^7}\cdot \sec^2\theta\, d\theta$$

$$\boxed{\begin{aligned} u &= \sin x \\ du &= \cos x\, dx \end{aligned}}$$

$$= \int \frac{1}{\left(\sqrt{\sec^2\theta}\right)^7}\cdot \sec^2\theta\, d\theta = \int \frac{\cancel{\sec^2\theta}}{\underset{\sec^5\theta}{\cancel{\sec^7\theta}}}\,d\theta$$

$$\boxed{\begin{aligned} u &= \tan\theta \\ du &= \sec^2\theta\, d\theta \end{aligned}}$$



$$= \int \cos^5\theta\, d\theta = \int \cos^4\theta\, \cos\theta\, d\theta = \int (\cos^2\theta)^2 \cos\theta\, d\theta$$

$$= \int (1-\sin^2\theta)^2 \cos\theta\, d\theta = \int (1-w^2)^2\, dw = \int 1 - 2w^2 + w^4\, dw$$

$$\boxed{\begin{aligned} w &= \sin\theta \\ dw &= \cos\theta\, d\theta \end{aligned}}$$

$$= w - \frac{2}{3}w^3 + \frac{w^5}{5} + C = \sin\theta - \frac{2}{3}\sin^3\theta + \frac{\sin^5\theta}{5} + C$$

$$= \frac{u}{\sqrt{1+u^2}} - \frac{2}{3}\left[\frac{u}{\sqrt{1+u^2}}\right]^3 + \frac{1}{5}\left[\frac{u}{\sqrt{1+u^2}}\right]^5 + C$$

$$= \boxed{\frac{\sin x}{\sqrt{1+\sin^2 x}} - \frac{2}{3}\left(\frac{\sin^3 x}{(1+\sin^2 x)^{3/2}}\right) + \frac{1}{5}\left(\frac{\sin^5 x}{(1+\sin^2 x)^{5/2}}\right) + C}$$

2. (Continued) Evaluate the following integral. Simplify.

Trig. Sub.

(b) $\int_{-1}^{0} x \arcsin x \, dx$ $= \frac{x^2}{2}\arcsin x \Big|_{-1}^{0} - \frac{1}{2}\int_{-1}^{0} \frac{x^2}{\sqrt{1-x^2}}\,dx$

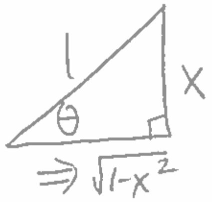
$$\boxed{\begin{array}{ll} u=\arcsin x & dv = x\,dx \\ du = \frac{1}{\sqrt{1-x^2}}dx & v=\frac{x^2}{2} \end{array}}$$

$$= \frac{x^2}{2}\arcsin x\Big|_{-1}^{0} - \frac{1}{2}\int_{x=-1}^{x=0} \frac{\sin^2\theta}{\underbrace{\sqrt{1-\sin^2\theta}}_{\cancel{\cos\theta}}} \cdot \cancel{\cos\theta}\, d\theta$$

$$\boxed{\begin{array}{l} x=\sin\theta \\ dx = \cos\theta\, d\theta \end{array}} \Rightarrow \theta = \arcsin x$$

$$= \frac{x^2}{2}\arcsin x\Big|_{-1}^{0} - \frac{1}{2}\int_{x=-1}^{x=0} \sin^2\theta \, d\theta$$

Half-Angle



$$= \frac{x^2}{2}\arcsin x\Big|_{-1}^{0} - \frac{1}{2}\int_{x=-1}^{x=0} \frac{1-\cos(2\theta)}{2}\, d\theta$$

$$= \frac{x^2}{2}\arcsin x\Big|_{-1}^{0} - \frac{1}{4}\left[\theta - \frac{\overset{\cancel{2}\sin\theta\cos\theta}{\sin(2\theta)}}{\cancel{2}}\right]\Bigg|_{x=-1}^{x=0.}$$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{4}\arcsin x + \frac{1}{4}x\cdot\sqrt{1-x^2}\,\Bigg|_{-1}^{0}$$

$$= 0 - \cancel{\tfrac{1}{4}\arcsin 0} \xrightarrow{} 0 + 0 - \left(\frac{1}{2}\arcsin(-1) - \tfrac{1}{4}\arcsin(-1) + 0\right)$$

$$= -\frac{1}{2}\left(-\frac{\pi}{2}\right) + \frac{1}{4}\left(-\pi/2\right)$$

$$= \frac{\pi}{4} - \pi/8 = \boxed{\frac{\pi}{8}}$$

3. [40 Points] For each of the following **improper integrals**, determine whether it converges or diverges. If it converges, find its value. Simplify.

(a) $\int_0^{e^3} \frac{1}{x\,[9+(\ln x)^2]}\,dx = \lim_{t\to0^+} \int_t^{e^3} \frac{1}{x(9+(\ln x)^2)}\,dx = \lim_{t\to0^+} \int_{\ln t}^{3} \frac{1}{9+u^2}\,du$

$\boxed{\begin{array}{l} u=\ln x \\ du=\frac{1}{x}\,dx \end{array}}$

$= \lim_{t\to0^+} \frac{1}{3}\arctan\left(\frac{u}{3}\right)\Big|_{\ln t}^{3}$

$\boxed{\begin{array}{l} x=t \Rightarrow u=\ln t \\ x=e^3 \Rightarrow u=\cancel{\ln} e^{3}=3 \end{array}}$

$= \lim_{t\to0^+} \frac{1}{3}\left[\cancelto{\pi/4}{\arctan 1} - \cancelto{(-\pi/2)}{\arctan\left(\frac{\cancelto{-\infty}{\ln t}}{3}\right)}\right]$

$= \frac{1}{3}\left[\frac{\pi}{4} \overset{--}{+} \frac{\pi}{2}\right] = \cancel{\frac{1}{3}}\left[\frac{3\pi}{4}\right] = \boxed{\frac{\pi}{4}}$ Converges.

(b) $\int_0^{e} \frac{\ln x}{\sqrt{x}}\,dx = \lim_{t\to0^+} \int_t^{e} \frac{\ln x}{\sqrt{x}}\,dx \overset{IBP}{=} \lim_{t\to0^+} 2\sqrt{x}\ln x\Big|_t^{e} - \int_t^{e} \frac{2}{\sqrt{x}}\,dx$

$\boxed{\begin{array}{ll} u=\ln x & dv=x^{-1/2}dx \\ du=\frac{1}{x} & v=2\sqrt{x} \end{array}}$

$= \lim_{t\to0^+} 2\sqrt{x}\ln x\Big|_t^{e} - 4\sqrt{x}\Big|_t^{e}$

$= \lim_{t\to0^+} 2\sqrt{e}\,\cancelto{1}{\ln e} - \cancelto{0}{\overset{0\cdot(-\infty)}{2\sqrt{t}\ln t}} - \left(4\sqrt{e} - \cancelto{0}{4\sqrt{t}}\right)$
(*) L'H.

$\overset{(*)}{=} 2\sqrt{e} - 0 - 4\sqrt{e}$

$= \boxed{-2\sqrt{e}}$ Converges.

(*) $\lim_{t\to0^+} \overset{0\cdot(-\infty)}{\sqrt{t}\ln t} = \lim_{t\to0^+} \frac{\ln t}{1/\sqrt{t}} \overset{-\infty/\infty}{\underset{L'H}{=}} \lim_{t\to0^+} \frac{1/t}{-1/2t^{3/2}} = \lim_{t\to0^+} \frac{-2t^{3/2}}{t} = \lim_{t\to0^+} -\cancelto{0}{2\sqrt{t}} = 0$

3. (Continued) For each of the following **improper integrals**, determine whether it converges or diverges. If it converges, find its value. Simplify.

(c) $\displaystyle\int_1^2 \frac{2}{x^2-6x+8}\,dx = \int_1^2 \frac{2}{(x-4)(x-2)}\,dx = \lim_{t\to 2^-}\int_1^t \frac{2}{(x-4)(x-2)}\,dx$

PFD

$$\frac{2}{(x-4)(x-2)} = \frac{A}{x-4} + \frac{B}{x-2}$$

$$2 = A(x-2)+B(x-4)$$

$$=(A+B)x-2A-4B$$

- $A+B=0 \Rightarrow B=-A$
- $-2A-4B=2$ ✓

$$-2A+4A=2$$

$$2A=2$$

$$A=1 \Rightarrow B=-1$$

$$\overset{\text{PFD}}{=} \lim_{t\to 2^-}\int_1^t \frac{1}{x-4} - \frac{1}{x-2}\,dx$$

$$= \lim_{t\to 2^-} \ln|x-4| - \ln|x-2| \Big|_1^t$$

$$= \lim_{t\to 2^-} \overset{\ln 2}{\ln|\overset{2}{\cancel{t}}-4|} - \overset{0^+\to -\infty}{\cancel{\ln|t-2|}} - (\ln 3 - \overset{0}{\cancel{\ln 1}})$$

$$= \ln 2 - (-\infty) - \ln 3 = \boxed{+\infty}$$ Diverges.

(d) $\displaystyle\int_5^\infty \frac{1}{x^2-6x+13}\,dx = \lim_{t\to\infty}\int_5^t \frac{1}{x^2-6x+13}\,dx$

Complete Square.

$$= \lim_{t\to\infty}\int_5^t \frac{1}{(x-3)^2+4}\,dx$$

$$\boxed{\begin{array}{l} u=x-3 \\ du=dx \end{array}}$$

$$\boxed{\begin{array}{l} x=5 \Rightarrow u=2 \\ x=t \Rightarrow u=t-3 \end{array}}$$

$$= \lim_{t\to\infty}\int_2^{t-3} \frac{1}{u^2+4}\,du = \lim_{t\to\infty} \frac{1}{2}\arctan\left(\frac{u}{2}\right)\Big|_2^{t-3}$$

$$= \lim_{t\to\infty} \frac{1}{2}\left[\overset{\pi/2}{\cancel{\arctan\left(\frac{t-3}{2}\right)}} - \overset{\pi/4}{\cancel{\arctan(1)}}\right]$$

$$= \frac{1}{2}\left[\pi/2 - \pi/4\right] = 1/2\left[\pi/4\right] = \boxed{\pi/8}$$ Converges.

4. [20 Points] Find the **sum** of each of the following series (which do converge). Simplify.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n\, 3^{2n-1}}{4^{2n+1}} = \overset{n=1}{-\frac{3}{4^3}} + \overset{n=2}{\frac{3^3}{4^5}} - \overset{n=3}{\frac{3^5}{4^7}} + \dots$

$a = \frac{-3}{64}$

$r = \frac{-3^2}{4^2} = \frac{-9}{16}$

Conv. b/c. $|r| = 9/16 < 1$

$$\text{Sum} = \frac{a}{1-r} = \frac{\frac{-3}{64}}{1-(-9/16)} = \frac{\frac{-3}{64}}{\frac{25}{16}} \overset{16/25}{} = \boxed{\frac{-3}{100}}$$

(b) $\sum_{n=0}^{\infty} \frac{(-1)^n\, (\ln 8)^n}{3^{n+1}\, n!} = \frac{1}{3} \sum_{n=0}^{\infty} \frac{\left(\frac{-\ln 8}{3}\right)^n}{n!} = \frac{1}{3} e^{\frac{-\ln 8}{3}} = \frac{1}{3} \cancel{e^{\ln}}(8^{-1/3})$

$$= \frac{1}{3} \cdot \frac{1}{\sqrt[3]{8}} = \frac{1}{3} \cdot \frac{1}{2} = \boxed{\frac{1}{6}}$$

(c) $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{(36)^n\, (2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n (\pi/6)^{2n}}{(2n+1)!} \cdot \left(\frac{\pi/6}{\pi/6}\right) = \frac{6}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n (\pi/6)^{2n+1}}{(2n+1)!} = \frac{6}{\pi} \cdot \cancelto{1/2}{\sin(\pi/6)}$

$$= \boxed{\frac{3}{\pi}}$$

(d) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots$ $\arctan(1) = \boxed{\frac{\pi}{4}}$

(e) $-\frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \frac{\pi^8}{8!} - \dots = \cancelto{-1}{\cos \pi} - 1 = \boxed{-2}$

$\cos \pi = 1 - \pi^2/2! + \pi^4/4! - \dots$

(f) $-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \dots = -(1 - 1/2 + 1/3 - 1/4 + \dots)$

$$= -\ln(1+1) = \boxed{-\ln 2}$$

5. [30 Points] In each case determine whether the given series is **absolutely convergent**, **conditionally convergent**, or **divergent**. Justify your answers.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n (n^3+7)}{n^7+3}$

A.S. $\sum_{n=1}^{\infty} \frac{n^3+7}{n^7+3} \approx \sum_{n=1}^{\infty} \frac{n^3}{n^7} = \sum_{n=1}^{\infty} \frac{1}{n^4}$ Conv. p-series $p=4>1$

$$\lim_{n\to\infty} \frac{\frac{n^3+7}{n^7+3}}{\frac{1}{n^4}} = \lim_{n\to\infty} \frac{n^7+7n^4}{n^7+3} \frac{(1/n^7)}{(1/n^7)} = \lim_{n\to\infty} \frac{1+\cancel{7/n^3}^{\,0}}{1+\cancel{3/n^7}_{\,0}} = 1$$ Finite, Non-zero.

$\Rightarrow$ A.S. also converges by LCT

$\Rightarrow$ $\boxed{\text{A.C.}}$ (by definition)

(b) $\sum_{n=1}^{\infty} (-1)^n \frac{n+1}{n^2}$

A.S. $\sum_{n=1}^{\infty} \frac{n+1}{n^2} \approx \sum_{n=1}^{\infty} \frac{n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n}$ Divergent Harmonic p-series $p=1$.

$$\lim_{n\to\infty} \frac{\frac{n+1}{n^2}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2+n}{n^2} = \lim_{n\to\infty} 1+\cancel{1/n}^{\,0} = 1$$ Finite, Non-Zero.

$\Rightarrow$ A.S. also Diverges by LCT

AST

① $b_n = \frac{n+1}{n^2} > 0$

② $\lim_{n\to\infty} \frac{n+1}{n^2} = 0$

③ $b_{n+1} < b_n$ Terms decreasing

because $f(x) = \frac{x+1}{x^2}$ has

$$f'(x) = \frac{x^2(1)-(x+1)(2x)}{x^4} = \frac{\overset{-x^2}{x^2-2x^2}-2x}{x^4} = \frac{-(x^2+2x)}{x^4} < 0 \text{ for } x>0.$$

O.S. Converges by AST

$\boxed{\text{C.C.}}$

5. (Continued) In each case determine whether the given series is **absolutely convergent**, **conditionally convergent**, or **divergent**. Justify your answers.

(c) $\sum_{n=1}^{\infty} \frac{(-1)^n \arctan(n^2)}{n^2+1}$ $\curvearrowright$ A.S. $\sum_{n=1}^{\infty} \frac{\arctan(n^2)}{n^2+1}$

Bound Terms

$$\frac{\arctan(n^2)}{n^2+1} \le \frac{\pi/2}{n^2+1} \le \frac{\pi/2}{n^2}$$

and $\frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$ is Convergent as a Constant Multiple of Convergent p-Series $p=2>1$

$\Rightarrow$ A.S Converges by CT

$\Rightarrow$ $\boxed{A.C.}$

(d) $\sum_{n=1}^{\infty} \arctan\left(\frac{n^2}{n^2+1}\right)$ Diverges by n^{th} Term Divergence Test b/c

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \arctan\left(\frac{n^2}{n^2+1}\right)$$

$$= \arctan\left[\lim_{n\to\infty} \frac{n^2}{n^2+1}\left(\frac{1/n^2}{1/n^2}\right)\right]$$

$$= \arctan\left[\lim_{n\to\infty} \frac{1}{1+\underbrace{1/n^2}_{\to 0}}\right] \to 1$$

$$= \arctan 1 = \pi/4 \neq 0.$$

5. (Continued) Determine whether the given series is **absolutely convergent**, **conditionally convergent**, or **divergent**. Justify your answers.

(e) $\sum_{n=1}^{\infty} \frac{(-1)^n\,(3n)!\,\ln n}{(n!)^2\,2^{4n}\,n^n}$

Ratio Test

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{\cancel{(-1)^{n+1}}\,(3(n+1))!\,\ln(n+1)}{((n+1)!)^2\,2^{4(n+1)}\,(n+1)^{n+1}}}{\dfrac{\cancel{(-1)^{n}}\,(3n)!\,\ln n}{(n!)^2\,2^{4n}\cdot n^n}}\right| = \lim_{n\to\infty}\frac{(3n+3)!}{(3n)!}\overset{\text{see}(*)\nearrow 1}{\cancel{\left(\frac{\ln(n+1)}{\ln n}\right)}}\cdot\frac{(n!)^2}{((n+1)!)^2}\left(\frac{\cancel{2^{4n}}}{\underbrace{2^{4n+4}}_{\cancel{2^{4n}}\cdot 2^4}}\right)\cdot\frac{n^n}{(n+1)^{n+1}}$$

$$= \lim_{n\to\infty}\frac{\overset{3(\cancel{n+1})}{(3n+3)}(3n+2)(3n+1)\cancel{(3n)!}}{\cancel{(3n)!}}\cdot(1)\cdot\frac{\cancel{n!\,n!}}{\underbrace{(n+1)^2}\cancel{(n!)^2}}\cdot\frac{1}{16}\cdot\underbrace{\cancel{\frac{n^n}{(n+1)^n}}}_{\swarrow\ 1/e}\cdot\cancel{\frac{1}{n+1}}$$

$$= \lim_{n\to\infty} 3\cdot\left(\frac{3n+2}{n+1}\right)\left(\frac{3n+1}{n+1}\right)\cdot\frac{1}{16}\cdot\frac{1}{e}$$

$$= \lim_{n\to\infty}\frac{3}{16e}\overset{3}{\left(\cancel{\frac{3+\overset{\nearrow 0}{2/n}}{1+1/n}}\right)}\cdot\overset{3}{\left(\cancel{\frac{3+\overset{\nearrow 0}{1/n}}{1+1/n}}\right)} = \frac{27}{16e} < 1$$

O.S. $\boxed{\text{Converges Absolutely}}$ by R.T.

$\boxed{\text{A.C.}}$

$$(*)\ \lim_{n\to\infty}\frac{\ln(n+1)}{\ln n}\overset{\infty/\infty}{=}\lim_{x\to\infty}\frac{\ln(x+1)}{\ln x}\overset{\infty/\infty}{\underset{L'H}{=}}\lim_{x\to\infty}\frac{\frac{1}{x+1}}{\frac{1}{x}}=\lim_{x\to\infty}\frac{x}{x+1}\overset{\infty/\infty}{\underset{L'H}{=}}\lim_{x\to\infty}\frac{1}{1}=1$$

$$\text{OR} = \lim_{x\to\infty}\frac{1}{1+\cancel{1/x}_{\infty}} = 1$$

6. [12 Points] Find the **Interval** and **Radius** of Convergence for the following power series. Analyze carefully and with full justification.

$$\sum_{n=1}^{\infty} \frac{(-1)^n\,(3x+1)^n}{(n+7)\cdot 7^n}$$

Ratio Test

$$L = \lim_{n\to\infty} \left| \frac{\dfrac{\cancel{(-1)^{n+1}}\,(3x+1)^{n+1}}{(n+8)\;7^{n+1}}}{\dfrac{\cancel{(-1)^{n}}\,(3x+1)^{n}}{(n+7)\cdot 7^{n}}} \right| = \lim_{n\to\infty} \left| \frac{(3x+1)^{n+1}}{(3x+1)^{n}} \right| \overset{1}{\cancel{\left(\frac{n+7}{n+8}\right)}} \frac{7^n}{7^{n+1}} = \frac{|3x+1|}{7} < 1$$

Converges by R.T. when

$$|3x+1| < 7 \qquad -7 < 3x+1 < 7$$

$$\begin{array}{ccccc} -7 & < & 3x+1 & < & 7 \\ -1 & & -1 & & -1 \end{array}$$

$$-8 < 3x < 6$$

$$-\tfrac{8}{3} < x < 2$$

Endpoints:

$x=2$ O.S. becomes $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(3(2)+1)^n}{(n+7)\cdot 7^n} = \sum_{n=1}^{\infty} \frac{(-1)^n \cancel{7^n}}{(n+7)\cancel{7^n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n+7}$ Converges by AST.

① $b_n = \frac{1}{n+7} > 0$

② $\displaystyle\lim_{n\to\infty} \frac{1}{n+7} = 0$

③ $b_{n+1} < b_n$

$b_{n+1} = \frac{1}{n+8} < \frac{1}{n+7} = b_n$ ✓

$x = -8/3$ O.S. becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n\left(3\left(-\tfrac{8}{3}\right)+1\right)^n}{(n+7)\,7^n} = \sum_{n=1}^{\infty} \frac{\overbrace{(-1)^n(-7)^n}^{\overbrace{(-1)^n 7^n}^{(-1)^{2n}}}}{(n+7)\cancel{7^n}} = \sum_{n=1}^{\infty} \frac{1}{n+7} \approx \sum_{n=1}^{\infty} \frac{1}{n}$$

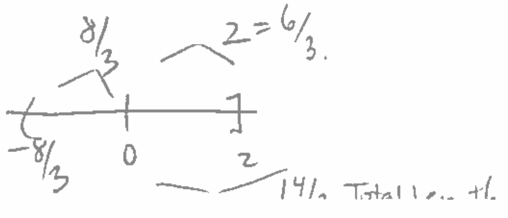
Div. Harmonic
$p=1$.

$$\lim_{n\to\infty} \frac{\frac{1}{n+7}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n\left(\frac{1}{n}\right)}{n+7\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{1}{1+\underset{\to 0}{\cancel{\tfrac{7}{n}}}} = 1$$ finite + Non-zero

Series Diverges by LCT

Finally, $I = \boxed{\left(-\tfrac{8}{3},\, 2\right]}$

$R = \boxed{\tfrac{7}{3}}$



7. [10 Points] (a) Use MacLaurin series to **Estimate** $\int_0^1 x\sin\left(x^2\right)\,dx$ with error less than $\frac{1}{1000}$. Please analyze with detail and justify carefully. Simplify.

$$\int_0^1 x\sin(x^2)dx = \int_0^1 x \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{\overset{\downarrow 4n+2}{2n+1}}}{(2n+1)!}\,dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(2n+1)!}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+4}}{(2n+1)!\,(4n+4)} \Bigg|_0^1 = \frac{x^4}{4} - \frac{x^8}{3!\cdot 8} + \frac{x^{12}}{5!\cdot(12)} - \cdots \Bigg|_0^1$$

$$= \frac{1}{4} - \frac{1}{48} + \frac{1}{1440} - \cdots \cancelto{0}{-(0-0+0-\ldots)}$$

$$\approx \frac{1}{4} - \frac{1}{48} = \boxed{\frac{11}{48}} \leftarrow \text{Estimate}$$

Using $\boxed{\text{ASET}}$ we can estimate the full sum using only the first two terms with error <u>at most</u> the absolute value of the first neglected term.

Here that is $\frac{1}{1440} < \frac{1}{1000}$ as desired.

(b) Estimate $\frac{1}{\sqrt{e}}$ with error less than $\frac{1}{100}$. Justify in words that your error is indeed less than $\frac{1}{100}$.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\frac{1}{\sqrt{e}} = e^{-1/2} = 1 - \frac{1}{2} + \frac{(-\frac{1}{2})^2}{2!} + \frac{(-\frac{1}{2})^3}{3!} + \frac{(-\frac{1}{2})^4}{4!} + \cdots$$

$$= 1 - \frac{1}{2} + \frac{1}{8} - \frac{1}{48} + \frac{1}{384} - \cdots$$

$$\approx 1 - \frac{1}{2} + \frac{1}{8} - \frac{1}{48} = \frac{48}{48} - \frac{24}{48} + \frac{6}{48} - \frac{1}{48} = \boxed{\frac{29}{48}} \leftarrow \text{Estimate}$$

Using ASET, we can use the first 4 terms to estimate the full Sum as $\frac{29}{48}$ with Error at most $\frac{1}{384} < \frac{1}{100}$ as desired.

8. [8 Points] For each of the following functions, find the MacLaurin Series and, then **State** the Radius of Convergence.

(a) $f(x) = \sinh x$ $= \frac{e^x - e^{-x}}{2} = \frac{1}{2}\left[e^x - e^{-x}\right] = \frac{1}{2}\left[\cancel{1}+x+\cancel{\frac{x^2}{2!}}+\frac{x^3}{3!}+\cdots - \left(\cancel{1}-x+\cancel{\frac{x^2}{2!}}-\frac{x^3}{3!}+\cdots\right)\right]$

$$= \cancel{\tfrac{1}{2}}\left[\cancel{2}x+\frac{\cancel{2}x^3}{3!}+\frac{\cancel{2}x^5}{5!}+\cdots\right] = x+\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots$$

$$= \boxed{\sum_{n=1}^{\infty}\frac{x^{2n+1}}{(2n+1)!}} \qquad \boxed{R=\infty}$$

"exponentials"

OR

$f(x)=\sinh x$ $\quad f(0)=\sinh 0=0$

$f'(x)=\cosh x$ $\quad f'(0)=\cosh 0=1$

$f''(x)=\sinh x$ $\quad f''(0)=0$

$f^{(3)}(x)=\cosh x$ $\quad f^{(3)}(0)=1$

$\vdots \qquad \vdots$

$$\cancel{f(0)}^{0}+f'(0)x+\cancel{\frac{f''(0)}{2!}}^{0}x^2+\frac{f'''(0)}{3!}x^3+\cdots$$

$$= x+\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots$$ Match! OR Run R.T. here.

$$= \sum_{n=1}^{\infty}\frac{x^{2n+1}}{(2n+1)!}$$

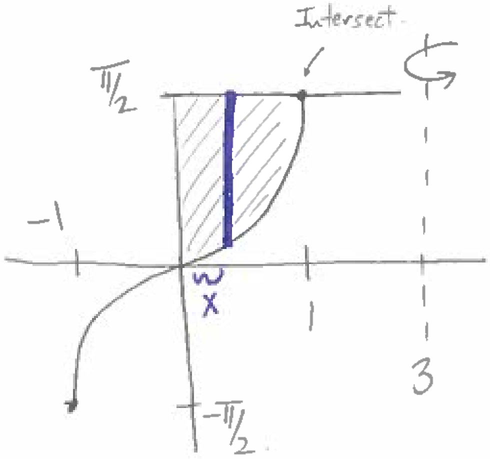
(b) $f(x) = \frac{1}{(1-x)^2}$. Hint: $\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right)$ Differentiate $\frac{1}{1-x}$.

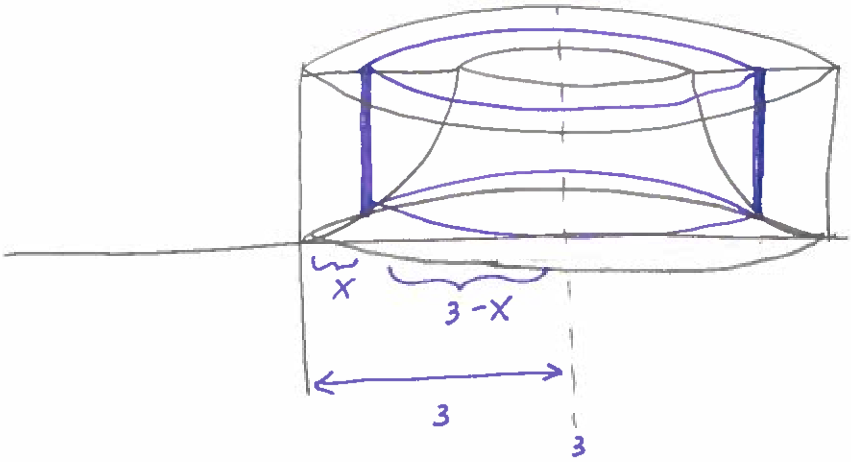
$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\underbrace{\left(\sum_{n=0}^{\infty}x^n\right)}_{\substack{R=1\\ \text{Conv. for}\\ |x|<1}} = \boxed{\sum_{n=0}^{\infty}nx^{n-1}}$$

$\boxed{R=1}$ still

9. [18 Points]

(a) Consider the region bounded by $y = \arcsin x$, $y = \frac{\pi}{2}$, and $x = 0$. Rotate the region about the vertical line $\boxed{x = 3}$. Set-up, **BUT DO NOT EVALUATE!!**, the integral to compute the volume of the resulting solid using the Cylindrical Shells Method. Sketch the solid, along with one of the approximating shells.



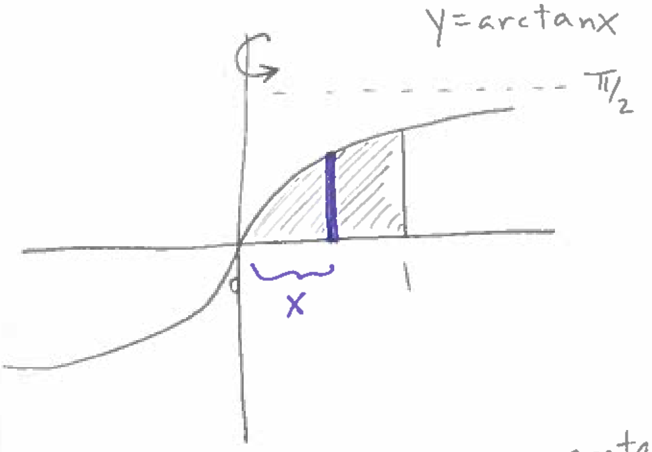


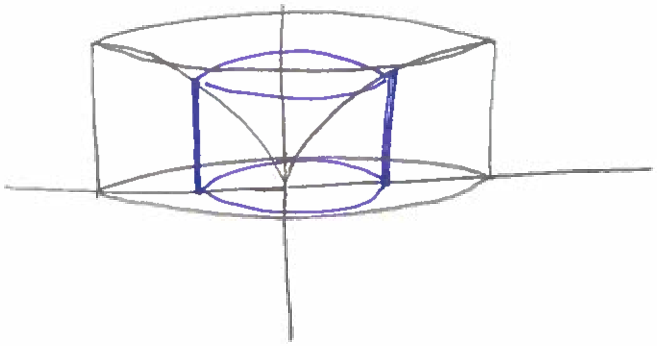
$$V = 2\pi \int_0^1 \text{Radius} \cdot \text{Height} \, dx$$

$$= \boxed{2\pi \int_0^1 (3-x)\left(\pi/2 - \arcsin x\right) dx}$$

9. (Continued)

(b) Consider the region bounded by $y = \arctan x$, $y = 0$, $x = 0$ and $x = 1$. Rotate the region about the vertical line $\boxed{y\text{-axis}}$. **COMPUTE** the volume of the resulting solid using the Cylindrical Shells Method. Sketch the solid, along with one of the approximating shells.





$$V = 2\pi\int_0^1 \overset{x}{\cancel{\text{Radius}}}\cdot\overset{\arctan x}{\cancel{\text{Height}}}\,dx$$

$$= 2\pi\int_0^1 x\arctan x\,dx = 2\pi\left[\frac{x^2}{2}\arctan x\Big|_0^1 - \frac{1}{2}\int_0^1 \frac{x^2 \overset{+1-1}{}}{1+x^2}\,dx\right]$$

slip-in
slip-out

$\boxed{\begin{array}{ll} u=\arctan x & dv = x\,dx \\ du = \dfrac{1}{1+x^2}dx & v = \dfrac{x^2}{2}\end{array}}$

$$= 2\pi\left[\frac{x^2}{2}\arctan x\Big|_0^1 - \frac{1}{2}\int_0^1 \frac{\cancel{x^2+1}}{\cancel{x^2+1}} - \frac{1}{x^2+1}\,dx\right]$$

$$= \cancel{2}\pi\left[\frac{x^2}{\cancel{2}}\arctan x - \frac{1}{\cancel{2}}x + \frac{1}{\cancel{2}}\arctan x\right]\Big|_0^1$$

Cancel 2's.

$$= \pi\left[\overset{\pi/4}{\cancel{\arctan}\,1} - 1 + \overset{\pi/4}{\cancel{\arctan}\,1} - \left(0 - 0 + \overset{0}{\cancel{\arctan}\,0}\right)\right]$$

$$= \pi\left[\pi/2 - 1\right] = \boxed{\frac{\pi^2}{2} - \pi}$$

10. [14 Points]

Consider the Parametric Curve represented by $x = \ln t + \ln(1-t^2)$ and $y = \sqrt{8}\arcsin t$.

COMPUTE the **arclength** of this parametric curve for $\frac{1}{4} \le t \le \frac{1}{2}$. **Show** that the answer simplifies to $\boxed{\ln\left(\frac{5}{2}\right)}$

$$\frac{dx}{dt} = \frac{1}{t} - \frac{2t}{1-t^2} \qquad \frac{dy}{dt} = \frac{\sqrt{8}}{\sqrt{1-t^2}}$$

$$L = \int_{1/4}^{1/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_{1/4}^{1/2} \sqrt{\underline{\left(\frac{1}{t} - \frac{2t}{1-t^2}\right)^2} + \left(\frac{\sqrt{8}}{\sqrt{1-t^2}}\right)^2}\, dt$$

$$= \int_{1/4}^{1/2} \sqrt{\frac{1}{t^2} - \frac{4}{1-t^2} + \frac{4t^2}{(1-t^2)^2} + \frac{8}{1-t^2}}\, dt = \int_{1/4}^{1/2} \sqrt{\frac{1}{t^2} + \frac{4}{1-t^2} + \frac{4t^2}{(1-t^2)^2}}\, dt$$

$$= \int_{1/4}^{1/2} \sqrt{\underline{\left(1/t + \frac{2t}{1-t^2}\right)^2}}\, dt = \int_{1/4}^{1/2} \frac{1}{t} + \frac{2t}{1-t^2}\, dt = \ln|t| - \ln|1-t^2| \Big|_{1/4}^{1/2}$$

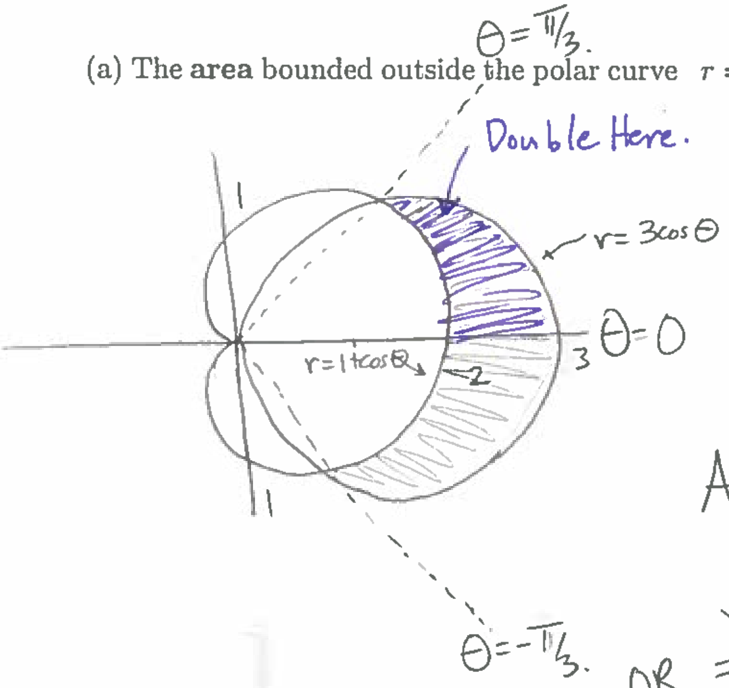
$$= \left(\ln\left(\frac{1}{2}\right) - \ln\left(\frac{3}{4}\right)\right) - \left(\ln\left(\frac{1}{4}\right) - \ln\left(\frac{15}{16}\right)\right) = \ln\left(\frac{1/2}{3/4}\right)^{\nearrow 4/3} - \ln\left(\frac{1/4}{15/16}\right)^{\nearrow 16/15}$$

$$= \ln\left(\frac{2}{3}\right) - \ln\left(\frac{4}{15}\right) = \ln\left(\frac{\frac{2}{3}}{\frac{4}{15}}\right)^{\nearrow 15/4} = \boxed{\ln\left(\frac{5}{2}\right)}$$

11. [18 Points] For each of the following problems, do the following **two** things:

1. Sketch the Polar curves and shade the described bounded region.
2. Set-Up but **DO NOT EVALUATE** the Integral representing the area of the described bounded region.

(a) The **area** bounded outside the polar curve $r = 1+\cos\theta$ and inside the polar curve $r = 3\cos\theta$.



Intersect $1+\cos\theta = 3\cos\theta$

$1 = 2\cos\theta$

$\frac{1}{2} = \cos\theta$

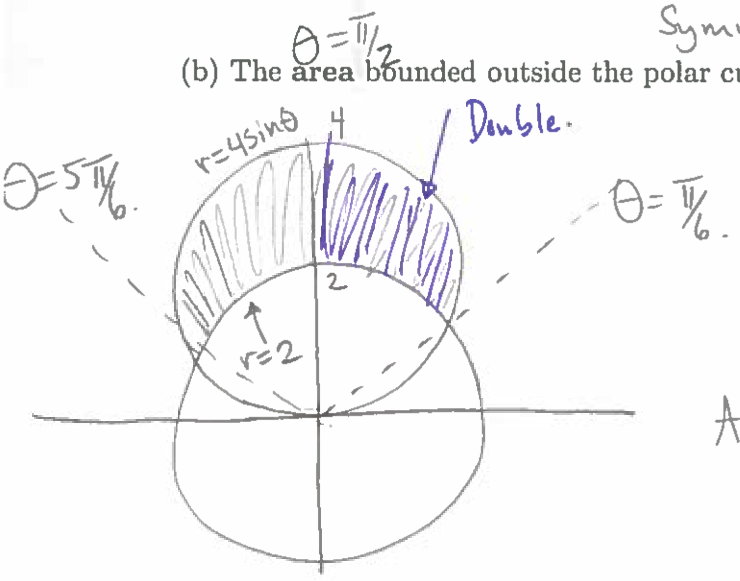
$\Longrightarrow \theta = \pm\pi/3$.

$$A = \frac{1}{2}\int_{-\pi/3}^{\pi/3} (\text{Outer Radius})^2 - (\text{Inner Radius})^2 \, d\theta$$

OR $= \cancel{2}\left[\cancel{\frac{1}{2}}\int_0^{\pi/3} (3\cos\theta)^2 - (1+\cos\theta)^2 \, d\theta\right]$

Double using Symmetry.

(b) The **area** bounded outside the polar curve $r = 2$ and inside the polar curve $r = 4\sin\theta$.



Intersect? $4\sin\theta = 2$

$\sin\theta = \frac{1}{2} \Rightarrow \theta = \pi/6, 5\pi/6$.

$$A = \frac{1}{2}\int_{\pi/6}^{5\pi/6} (\text{Outer Radius})^2 - (\text{Inner Radius})^2 \, d\theta$$

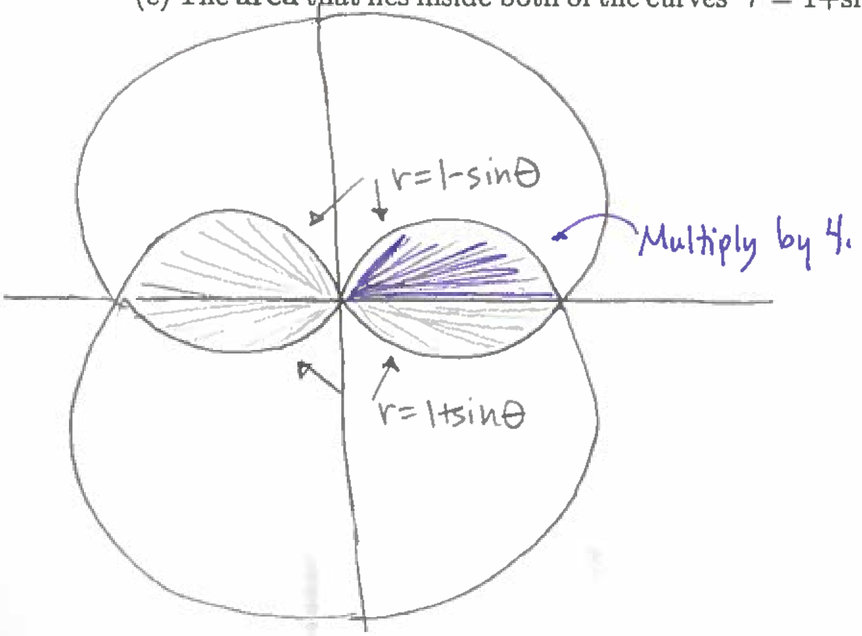
OR $= \cancel{2}\left[\cancel{\frac{1}{2}}\int_{\pi/6}^{\pi/2} (4\sin\theta)^2 - (2)^2 \, d\theta\right]$

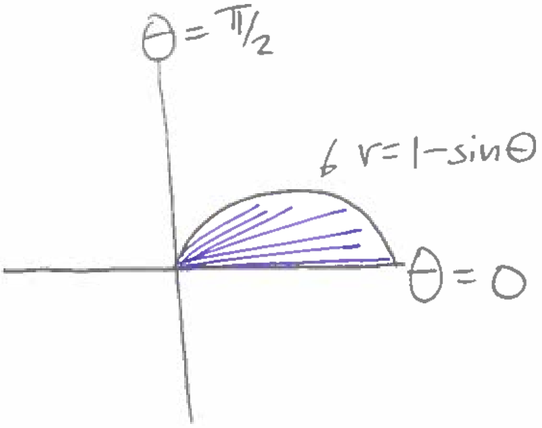
Double using Symmetry

11. (Continued) For the following problem, do the following **two** things:

1. Sketch the Polar curves and shade the described bounded region.
2. Set-Up but **DO NOT EVALUATE** the Integral representing the area of the described bounded region.

(c) The **area** that lies inside both of the curves $r = 1+\sin\theta$ and inside the polar curve $r = 1-\sin\theta$.





$$A=4\left[\frac{1}{2}\int_0^{\pi/2}(\text{Radius})^2\,d\theta\right]=4\cdot\left[\frac{1}{2}\int_0^{\pi/2}(1-\sin\theta)^2\,d\theta\right]$$

Multiples using Symmetry

OR $=4\left[\frac{1}{2}\int_{\pi/2}^{\pi}(1-\sin\theta)^2\,d\theta\right]\overset{\text{OR}}{=}4\left[\frac{1}{2}\int_{\pi}^{3\pi/2}(1+\sin\theta)^2\,d\theta\right]$

$$=4\left[\frac{1}{2}\int_{3\pi/2}^{2\pi}(1+\sin\theta)^2\,d\theta\right]$$

OR $=2\left[\frac{1}{2}\int_0^{\pi}(1-\sin\theta)^2\,d\theta\right]$

OR $=2\left[\frac{1}{2}\int_{\pi}^{2\pi}(1+\sin\theta)^2\,d\theta\right]$

Many Options Here.