# Sequences   Exam #2 Review Packet

Algebra

52. $\lim_{n\to\infty} \frac{1+n-7n^4}{3n^4+8n^3+9} \frac{(1/n^4)}{(1/n^4)} = \lim_{n\to\infty} \frac{\cancel{1/n^4}+\cancel{1/n^3}-7}{3+\cancel{8/n}+\cancel{9/n^4}} = \boxed{\frac{-7}{3}}$ Converges

OR L'H $= \lim_{x\to\infty} \frac{1+x-7x^4}{3x^4+8x^3+9} \underset{L'H}{=} \lim_{x\to\infty} \frac{1-28x^3}{12x^3+24x^2} \underset{L'H}{=} \lim_{x\to\infty} \frac{-84x^2}{36x^2+48x} \underset{L'H}{=} \lim_{x\to\infty} \frac{-168x}{72x+48}$

$\underset{L'H}{=} \lim_{x\to\infty} \frac{-168}{72} = \boxed{\frac{-7}{3}}$   Note: L'H Messier   $168 = 24\cdot 7$, $72 = 24\cdot 3$

53. $\lim_{n\to\infty} \frac{n^3}{(n+1)^3} = \lim_{n\to\infty}\left(\frac{n\,(1/n)}{n+1\,(1/n)}\right)^3 = \lim_{n\to\infty}\left(\frac{1}{1+\cancel{1/n}}\right)^3 = \boxed{1}$ Converges

54. $\lim_{n\to\infty}\left(\frac{n-5}{n}\right)^n \overset{1^\infty}{=} \lim_{x\to\infty}\left(\frac{x-5}{x}\right)^x = e^{\lim_{x\to\infty} \ln\left[\left(\frac{x-5}{x}\right)^x\right]} = e^{\lim_{x\to\infty} x\ln(1-5/x)}$ $\infty\cdot 0$

$= e^{\lim_{x\to\infty} \frac{\ln(1-5/x)}{1/x}} \underset{L'H}{=} e^{\lim_{x\to\infty} \frac{\frac{1}{1-5/x}(+5/x^2)}{-1/x^2}}$   check signs

$= \boxed{e^{-5}}$ Converges

55. $\lim_{n\to\infty} \frac{2^n}{n!}$

$\frac{2^n}{n!} = \frac{2}{n}\cdot\frac{2}{(n-1)}\cdot\frac{2}{(n-2)}\cdot 2 \cdots \cdot \frac{2}{4}\cdot\frac{2}{3}\cdot\frac{2}{2}\cdot\frac{2}{1}$

Use Squeeze Theorem   $\le \frac{2}{n}\cdot 1\cdot 1\cdot 1 \cdots 1\cdot 1\cdot 1\cdot 2 = \frac{4}{n}$

Bound   $0 \le \frac{2^n}{n!} \le 4/n$

$\lim_{n\to\infty} \frac{2^n}{n!} = \boxed{0}$ by Squeeze Theorem.   $0$   $n\to\infty$

56. $\lim_{n\to\infty} \frac{n!}{3^n} = \boxed{\infty}$ $\quad \frac{n!}{3^n} = \frac{n(n-1)(n-2)\cdots 4\cdot 3\cdot 2\cdot 1}{3\cdot 3\cdot 3 \cdots 3\cdot 3\cdot 3\cdot 3}$

$$\geq \frac{n}{3}\cdot 1\cdot 1 \cdots\cdots 1\cdot 1\cdot \frac{2}{3}\cdot\frac{1}{3} = \frac{2n}{27}$$

$$\lim_{n\to\infty}\frac{n!}{3^n} \geq \lim_{n\to\infty}\frac{2n}{27} \to \infty \quad \text{Diverges}$$

$\infty$

57. $\lim_{n\to\infty}\frac{(2n+3)!}{(2n+5)!} = \lim_{n\to\infty}\frac{\cancel{(2n+3)!}}{(2n+5)(2n+4)\cancel{(2n+3)!}} = \lim_{n\to\infty}\frac{1}{(2n+5)(2n+4)} = \boxed{0}$ Converges

$\infty$

58. $\lim_{n\to\infty}\arctan(n^2+1) = \boxed{\frac{\pi}{2}}$ Converges

59. $\lim_{n\to\infty}\frac{\sqrt{n}}{(\ln n)^2} \overset{\infty/\infty}{=} \lim_{x\to\infty}\frac{\sqrt{x}}{(\ln x)^2} \underset{L'H}{=} \lim_{x\to\infty}\frac{\frac{1}{2\sqrt{x}}}{2\ln x\cdot\frac{1}{x}} = \lim_{x\to\infty}\frac{x}{4\sqrt{x}\ln x} = \lim_{x\to\infty}\frac{\sqrt{x}}{4\ln x}$ $\infty/\infty$

$$\underset{L'H}{=} \lim_{x\to\infty}\frac{\frac{1}{2\sqrt{x}}}{4/x} = \lim_{x\to\infty}\frac{\sqrt{x}}{8} \to \infty = \boxed{\infty} \quad \text{Diverges}$$

60. $\lim_{n\to\infty}(e^n+n)^{1/n} \overset{\infty^0}{=} \lim_{x\to\infty}(e^x+x)^{1/x} = e^{\lim_{x\to\infty}\ln[(e^x+x)^{1/x}]} = e^{\lim_{x\to\infty}\frac{\ln(e^x+x)}{x}}$ $\infty/\infty$

$$\underset{L'H}{=} e^{\lim_{n\to\infty}\frac{\frac{1}{e^x+x}\cdot(e^x+1)}{1}} = e^{\lim_{x\to\infty}\frac{e^x+1}{e^x+x}} \underset{L'H}{=} e^{\lim_{x\to\infty}\frac{e^x}{e^x+1}}$$

$$\underset{L'H}{=} e^{\lim_{x\to\infty}\frac{\cancel{e^x}}{\cancel{e^x}}} = \boxed{e} \quad \text{Converges}$$

61. $\lim_{n\to\infty} n^{1/n} = \lim_{x\to\infty}x^{1/x} = e^{\lim_{x\to\infty}\ln(x^{1/x})} = e^{\lim_{x\to\infty}\frac{\ln x}{x}} \underset{L'H}{=} e^{\lim_{x\to\infty}\frac{1/x}{1}} = e^0 = \boxed{1}$

Converges

62. $\lim_{n\to\infty} n\sin\left(\frac{1}{n}\right) \overset{\infty\cdot 0}{=} \lim_{x\to\infty} x\sin\left(\frac{1}{x}\right) = \lim_{x\to\infty} \frac{\sin(1/x)}{1/x} \overset{0/0}{\underset{L'H}{=}} \lim_{x\to\infty} \frac{\cancel{\cos(1/x)}\cdot\cancel{(-1/x^2)}}{\cancel{-1/x^2}} = \boxed{1}$

Converges

## Series Sums

63. $\sum_{n=1}^{\infty} \frac{2^n+3^n}{6^n} = \sum_{n=1}^{\infty} \frac{2^n}{6^n} + \sum_{n=1}^{\infty} \frac{3^n}{6^n} = \sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n + \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n$

$\frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \ldots$

Convergent Geometric

$|r| = 1/3 < 1$

$a = 1/3 \quad r = \frac{1}{3}$

$SUM = \frac{a}{1-r} = \frac{1/3}{1-1/3} = \frac{1/3}{2/3} = 1/2$

$\frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \ldots$

Convergent Geometric

$|r| = \frac{1}{2} < 1$

$a = 1/2 \quad r = \frac{1}{2}$

$SUM = \frac{a}{1-r} = \frac{\frac{1}{2}}{1-1/2} = \frac{1/2}{1/2} = 1$

$+$

Sum of 2 Convergent Series is Convergent

$SUM = \boxed{3/2}$

64. $\sum_{n=0}^{\infty} \frac{1}{4^n} - \frac{1}{7^n} = \sum_{n=0}^{\infty} \frac{1}{4^n} - \sum_{n=0}^{\infty} \frac{1}{7^n}$

$1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \ldots$

Conv. Geometric

$|r| = 1/4$

$a = 1, r = 1/4$

$SUM = \frac{a}{1-r} = \frac{1}{1-1/4} = \frac{1}{3/4} = 4/3$

$-$

$1 + \frac{1}{7} + \left(\frac{1}{7}\right)^2 + \ldots$

Conv. Geometric

$|r| = 1/7$

$a = 1, r = \frac{1}{7}$

$SUM = \frac{a}{1-r} = \frac{1}{1-1/7} = \frac{1}{6/7} = 7/6$

$\frac{4}{3} - \frac{7}{6} = \frac{8}{6} - \frac{7}{6} = \boxed{\frac{1}{6}}$

Difference of Two Convergent Series is Convergent

65. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}2^{n-1}}{3^{n+1}} = \overset{n=1}{\frac{2^0}{3^2}} - \overset{n=2}{\frac{2^1}{3^3}} + \frac{2^2}{3^4} - \dots$

$a = \frac{1}{9} \quad r = \frac{-2}{3} \qquad \text{SUM} = \frac{a}{1-r} = \frac{1/9}{1-(-2/3)} = \frac{1/9}{5/3} \overset{3/5}{=} \boxed{\frac{1}{15}}$

Convergent Geometric $|r| = 2/3 < 1$

66. $\displaystyle\sum_{n=1}^{\infty} \frac{3^{n+2}}{2^{4n-1}} = \frac{3^3}{2^3} + \frac{3^4}{2^7} + \frac{3^5}{2^{11}} + \dots$

$a = 27/8 \quad r = \frac{3}{2^4} = 3/16$

Convergent Geometric Series $|r| = 3/16 < 1 \qquad \text{SUM} = \frac{a}{1-r} = \frac{27/8}{1-3/16} = \frac{\frac{27}{8}}{\frac{13}{16}} \overset{16/13}{=} \boxed{\frac{54}{13}}$

67. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} - \dots$ Telescoping

$n^{th}$ Partial Sum $= S_n = 1 - \cancel{\frac{1}{\sqrt{2}}} + \cancel{\frac{1}{\sqrt{2}}} - \cancel{\frac{1}{\sqrt{3}}} + \cancel{\frac{1}{\sqrt{3}}} - \cancel{\frac{1}{\sqrt{4}}} + \dots + \cancel{\frac{1}{\sqrt{n-1}}} - \cancel{\frac{1}{\sqrt{n}}} + \cancel{\frac{1}{\sqrt{n}}} - \frac{1}{\sqrt{n+1}}$

$= 1 - \frac{1}{\sqrt{n+1}}$

Full Sum $\displaystyle\sum \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 - \overset{0}{\cancel{\frac{1}{\underset{\infty}{\sqrt{n+1}}}}} = \boxed{1}$

68. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3\cdot 2^n} = \frac{1}{3\cdot 2} - \frac{1}{3\cdot 2^2} + \frac{1}{3\cdot 2^3} - \dots$

$a = \frac{1}{6}, \quad r = \frac{-1}{2} \quad$ Convergent Geometric $|r| = |-1/2| = \frac{1}{2} < 1$

$\text{SUM} = \frac{a}{1-r} = \frac{1/6}{1-(-1/2)} = \frac{1/6}{3/2} \overset{2/3}{=} \boxed{\frac{1}{9}}$

69. $\displaystyle\sum_{n=1}^{\infty} e^{1/n} - e^{\frac{1}{n+1}} = e^1 - e^{1/2} + e^{1/2} - e^{1/3} + \dots$ Telescoping

$S_n = e^1 - \cancel{e^{1/2}} + \cancel{e^{1/2}} - \cancel{e^{1/3}} + \dots + \cancel{e^{1/n-1}} - \cancel{e^{1/n}} + \cancel{e^{1/n}} - e^{\frac{1}{n+1}}$

$= e - e^{1/n+1}$

Full Sum $= \displaystyle\lim_{n\to\infty} S_n = \lim_{n\to\infty} e - e^{\overset{0}{1/n+1}} = e - e^{0} \overset{1}{=} \boxed{e-1}$

70. $\sum_{n=1}^{\infty} \frac{4^n}{3^{2n-1}} = \frac{4}{3} + \frac{4^2}{3^3} + \frac{4^3}{3^5} + \dots$

$a = 4/3 \quad r = \frac{4}{3^2} = 4/9$

$\text{SUM} = \frac{a}{1-r} = \frac{4/3}{1-4/9} = \frac{4/3}{5/9} = \boxed{\frac{12}{5}}$

Convergent Geometric $|r| = 4/9 < 1$

71. $\sum_{n=1}^{\infty} \frac{1}{n^2+n} = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \dots$

PFD

$\left[\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}\right] n(n+1)$

$1 = A(n+1) + Bn$

$= (A+B)n + A$

- $A+B=0 \Rightarrow B=-1$
- $A=1$

$S_n = 1 - \cancel{\tfrac{1}{2}} + \cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}} + \dots + \cancel{\tfrac{1}{n-1}} - \cancel{\tfrac{1}{n}} + \cancel{\tfrac{1}{n}} - \frac{1}{n+1}$

$\text{Full Sum} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 - \cancel{\frac{1}{n+1}}^{\,0} = \boxed{1}$

72. $\sum_{n=1}^{\infty} \frac{(-1)^n 4^n}{9^{n-1}} = \frac{-4}{9^0} + \frac{4^2}{9} - \frac{4^3}{9^2} + \dots$

$a = -4 \quad r = -4/9$ Convergent Geometric $|r| = |-4/9| = 4/9 < 1$

$\text{SUM} = \frac{a}{1-r} = \frac{-4}{1-(-4/9)} = \frac{-4}{13/9} = \boxed{\frac{-36}{13}}$

## More Series

73. $\sum_{n=1}^{\infty} \frac{(-1)^n n}{2^n}$ $\quad L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{\cancel{(-1)^{n+1}}(n+1)}{2^{n+1}}}{\frac{\cancel{(-1)^n} n}{2^n}}\right| = \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{2^n}{2^{n+1}} = \lim_{n\to\infty} \left(\frac{1+\cancel{1/n}}{1}\right) \cdot \frac{1}{2} = \frac{1}{2} < 1$

O.S. A.C. by R.T.

$\Rightarrow$ $\boxed{\text{Converges}}$ by ACT

74. $\sum_{n=1}^{\infty} \frac{2n + \ln n}{n+2010}$ $\boxed{\text{Diverges}}$ by $n^{th}$ Term Divergence Test

$\lim_{n\to\infty} \frac{2n+\ln n}{n+2010} \frac{(1/n)}{(1/n)} = \lim_{n\to\infty} \frac{2 + \cancel{\frac{\ln n}{n}}^{\,0\,(*)}}{1 + \cancel{\frac{2010}{n}}_{\,0}} = 2 \neq 0$ OR L'H $\lim_{x\to\infty} \frac{2x + \ln x}{x + 2010} \overset{\infty/\infty}{=} \lim_{x\to\infty} \frac{2 + \cancel{1/x}^{\,0}}{1} = 2 \neq 0$

(*) $\lim_{x\to\infty} \frac{\ln x}{x} \underset{L'H}{\overset{\infty/\infty}{=}} \lim_{x\to\infty} \frac{\cancel{1/x}^{\,0}}{1} = 0$

75. $\sum_{n=1}^{\infty} \frac{e^n}{n^2}$ $L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{e^{n+1}}{(n+1)^2}}{\frac{e^n}{n^2}}\right| = \lim_{n\to\infty} \frac{\cancel{e^{n+1}}^{\,e}}{\cancel{e^n}} \cdot \frac{n^2}{(n+1)^2}$

$= \lim_{n\to\infty} e \cdot \left(\frac{n}{n+1}\,\frac{(1/n)}{(1/n)}\right)^2 = \lim_{n\to\infty} e\left(\frac{1}{1+\cancel{1/n}_{\to 0}}\right)^{2\to 1} = e > 1$

O.S. $\boxed{\text{Diverges}}$ by R.T.

OR

Diverges by $n^{th}$ Term Divergence Test because

$\lim_{n\to\infty} \frac{e^n}{n^2} \overset{\infty/\infty}{=} \lim_{x\to\infty} \frac{e^x}{x^2} \overset{\infty/\infty}{\underset{L'H}{=}} \lim_{x\to\infty} \frac{e^x}{2x} = \lim_{x\to\infty} \frac{e^x}{2} = \infty \neq 0$

76. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2+1}$ ACT not helpful b/c A.S. $= \sum \frac{n}{n^2+1} \approx \sum \frac{1}{n}$ Diverges Harmonic p-Series $=1$

AST ① $b_n = \frac{n}{n^2+1} > 0$

② $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n}{n^2+1}\,\frac{(1/n^2)}{(1/n^2)} = \lim_{n\to\infty} \frac{1/n^{\to 0}}{1+1/n^2_{\to 0}} = 0$

③ $b_{n+1} \le b_n$ $f(x) = \frac{x}{x^2+1}$ $f'(x) = \frac{(x^2+1)(1) - x(2x)}{(x^2+1)^2} = \frac{1-x^2}{\underset{+}{(x^2+1)^2}} < 0$ when $1-x^2<0$ when $x>1$

$\Rightarrow$ O.S. $\boxed{\text{Converges}}$ by AST

77. $\sum_{n=1}^{\infty} \frac{2^n n^2}{n!}$ $L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{2^{n+1}(n+1)^2}{(n+1)!}}{\frac{2^n n^2}{n!}}\right| = \lim_{n\to\infty} \frac{\cancel{2^{n+1}}^{\,2}}{\cancel{2^n}} \cdot \frac{(n+1)^2}{n^2} \cdot \frac{\cancel{n!}}{\underbrace{(n+1)!}_{(n+1)\cancel{n!}}}$

$= \lim_{n\to\infty} 2 \cdot \left(\frac{n+1}{n}\right)^2 \cdot \frac{1}{n+1} = \lim_{n\to\infty} 2\left(\frac{1+\cancel{1/n}_{\to 0}}{1}\right)^{2\to 1} \cdot \frac{1}{n+1}_{\to 0} = 0 < 1$

O.S. A.C. by R.T. $\Rightarrow$ $\boxed{\text{Converges}}$ by A.C.T.

Check 3 Conditions

78. $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ Related Function $f(x)=\frac{\ln x}{x^2}$

1. Continuous $x>0$
2. Positive $x>1$
3. Decreasing $f'(x)=\frac{x^2(1/x)-\ln x(2x)}{x^4}$

$= \frac{x-2x\ln x}{x^4}$

$= \frac{\cancel{x}(1-2\ln x)}{\cancel{x^4}}$

$= \frac{1-2\ln x}{x^3} < 0$

when $1-2\ln x<0$

$\frac{1}{2} < \ln x$

$e^{1/2} < x$ o.k.

$$\int_1^{\infty} \frac{\ln x}{x^2}dx = \lim_{t\to\infty}\int_1^t \frac{\ln x\,dx}{x^2} = \lim_{t\to\infty}\int_1^t \ln x \cdot x^{-2}dx$$

$\boxed{\begin{array}{ll} u=\ln x & dv=x^{-2}dx \\ du=\frac{1}{x}dx & v=-\frac{1}{x}\end{array}}$

$$=\lim_{t\to\infty} -\frac{\ln x}{x}\Big|_1^t + \int_1^t \frac{1}{x^2}dx$$

$$=\lim_{t\to\infty} -\frac{\ln x}{x}\Big|_1^t - \frac{1}{x}\Big|_1^t$$

$$=\lim_{t\to\infty} -\frac{\ln t}{t} + \cancel{\ln 1}^{\,0} - \cancel{\frac{1}{t}}^{\,0} + 1$$

$$\underset{L'H}{=} \lim_{t\to\infty} \frac{-1/t}{1}^{\,0} + 1 = \boxed{1} \text{ Integral Converges}$$

$\Rightarrow$ O.S. $\boxed{\text{Converges}}$ by I.T.

79. $\sum_{n=1}^{\infty} \frac{n^2+1}{2n^2\sqrt{n}+9} \approx \sum_{n=1}^{\infty} \frac{\cancel{n^2}}{\cancel{n^2}\sqrt{n}} = \sum \frac{1}{\sqrt{n}}$ Divergent p-series $p=\frac{1}{2}<1$

$$\lim_{n\to\infty} \frac{\frac{n^2+1}{2n^{5/2}+9}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{n^{5/2}+\sqrt{n}}{2n^{5/2}+9}\frac{(1/n^{5/2})}{(1/n^{5/2})} = \lim_{n\to\infty}\frac{1+\cancel{\frac{1}{n^2}}^{\,0}}{2+\cancel{\frac{9}{n^{5/2}}}^{\,0}} = \frac{1}{2}$$ Finite, Non-Zero

$\Rightarrow$ O.S. also $\boxed{\text{Diverges}}$ by LCT

80. $\sum_{n=1}^{\infty} \frac{\sqrt{n}+3}{4n^2-2} \approx \sum \frac{\sqrt{n}}{n^2} = \sum \frac{1}{n^{3/2}}$ Converges p-Series $p=3/2>1$

$$\lim_{n\to\infty} \frac{\frac{\sqrt{n}+3}{4n^2-2}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{n^2+3}{4n^2-2}\frac{(1/n^2)}{(1/n^2)} = \lim_{n\to\infty}\frac{1+\cancel{3/n^2}^{\,0}}{4-\cancel{2/n^2}^{\,0}} = \frac{1}{4}$$ Finite, Non-Zero

$\Rightarrow$ O.S. also $\boxed{\text{Converges}}$ by LCT

81. $\sum_{n=1}^{\infty} \frac{n^{19}+40n^6+4n^3+19}{4+17n^5+n^{20}} \approx \sum \frac{n^{19}}{n^{20}} = \sum \frac{1}{n}$ Diverges Harmonic p-series $p=1$

LCT $\lim_{n\to\infty} \frac{\frac{n^{19}+40n^6+4n^3+19}{4+17n^5+n^{20}}}{1/n} = \lim_{n\to\infty} \frac{n^{20}+40n^7+4n^4+19n}{4+17n^5+n^{20}} \frac{\left(\frac{1}{n^{20}}\right)}{\left(\frac{1}{n^{20}}\right)}$

$= \lim_{n\to\infty} \frac{1+\cancel{\frac{40}{n^{13}}}^{0}+\cancel{\frac{4}{n^{16}}}^{0}+\cancel{\frac{19}{n^{19}}}^{0}}{\cancel{\frac{4}{n^{20}}}_{0}+\cancel{\frac{17}{n^{15}}}_{0}+1} = 1$ Finite, Non-zero

$\Rightarrow$ O.S. also $\boxed{\text{Diverges}}$ by LCT

82. $\sum_{n=1}^{\infty} \frac{1}{n(\ln 2)^n}$

$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{1}{(n+1)(\ln 2)^{n+1}}}{\frac{1}{n(\ln 2)^n}}\right| = \lim_{n\to\infty} \frac{n}{n+1}\frac{(1/n)}{(1/n)} \frac{(\ln 2)^n}{(\ln 2)^{n+1}} = \lim_{n\to\infty} \cancel{\frac{1}{1+1/n}}^{1} \cdot \frac{1}{\ln 2}$

$= \frac{1}{\ln 2} > 1$

O.S. $\boxed{\text{Diverges}}$ by R.T.

$\ln 2 < 1$

$\ln e = 1$

$2 < e$

83. $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^2}$ Bound $\ln n \leq \sqrt{n}$

$\frac{1}{\ln n} \geq \frac{1}{\sqrt{n}}$

Bound Terms $\frac{1}{(\ln n)^2} \geq \frac{1}{(\sqrt{n})^2} = \frac{1}{n}$ and $\sum \frac{1}{n}$ Divergent Harmonic p-series $p=1$

$\Rightarrow$ O.S. $\boxed{\text{Diverges}}$ by CT

84. $\sum_{n=1}^{\infty} \frac{\ln n}{e^n}$ O.S. A.C. by R.T. $\Rightarrow$ $\boxed{\text{Converges}}$ by ACT

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{\ln(n+1)}{e^{n+1}}}{\frac{\ln n}{e^n}}\right| = \lim_{n\to\infty} \cancel{\frac{\ln(n+1)}{\ln n}}^{1\ (*)\ \text{L'H}} \cdot \frac{\cancel{e^n}}{\cancel{e^{n+1}}_{e}} = \frac{1}{e} < 1$

(*) $\lim_{x\to\infty} \frac{\ln(x+1)}{\ln x} \overset{\infty/\infty}{\underset{\text{L'H}}{=}} \lim_{x\to\infty} \frac{\frac{1}{x+1}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{x}{x+1} \overset{\infty/\infty}{\underset{\text{L'H}}{=}} \lim_{x\to\infty} \frac{1}{1} = 1$

85. $\sum_{n=2}^{\infty} \frac{1}{n\ln n}$ $f(x) = \frac{1}{x\ln x}$

① Positive $x>1$
② Continuous $x>1$
③ decreasing $f'(x) = \frac{-1}{(x\ln x)^2}\left(\cancel{x}\cdot\frac{1}{\cancel{x}} + \ln x(1)\right)$

$= \frac{-(1+\ln x)}{(x\ln x)^2} < 0$ when $1+\ln x > 0$

$\ln x > -1$

$x > e^{-1}$ o.k

$$\int_2^{\infty} \frac{1}{x\ln x}\,dx = \lim_{t\to\infty}\int_2^t \frac{1}{x\ln x}\,dx = \lim_{t\to\infty}\int_{\ln 2}^{\ln t}\frac{1}{u}\,du$$

$\boxed{\begin{array}{l} u=\ln x \\ du = \frac{1}{x}dx \end{array}}$ $\boxed{\begin{array}{l} x=2 \Rightarrow u=\ln 2 \\ x=t \Rightarrow u=\ln t \end{array}}$

$$= \lim_{t\to\infty} \ln|u|\Big|_{\ln 2}^{\ln t}$$

$$= \lim_{t\to\infty} \ln|\ln t| - \ln(\ln 2) = \infty \text{ Diverges}$$

$\Rightarrow$ O.S. $\boxed{\text{Diverges}}$ by I.T.

86. $\sum_{n=1}^{\infty} (-1)^n \frac{n}{3n+2}$

$$\lim_{n\to\infty} \frac{n}{3n+2}\frac{(1/n)}{(1/n)} = \lim_{n\to\infty}\frac{1}{3+2/n} = \frac{1}{3} \neq 0 \Rightarrow \lim_{n\to\infty}(-1)^n\frac{n}{3n+2} \neq 0 \text{ (DNE actually)}$$

$\Rightarrow$ O.S. Diverges by $n^{th}$ Term Divergence Test

87. $\sum_{n=1}^{\infty} ne^{-n^2}$

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{n+1}{e^{(n+1)^2}}}{\frac{n}{e^{n^2}}}\right| = \lim_{n\to\infty}\frac{n+1\,(1/n)\,\cancel{e^{n^2}}}{n\,(1/n)\,\underbrace{e^{n^2+2n+1}}_{\cancel{e^{n^2}}\cdot e^{2n}\cdot e}} = \lim_{n\to\infty}\frac{1+1/n}{1}\cdot\frac{1}{e^{2n}\cdot e} = 0 < 1$$

O.S. Converges Absolutely by R.T. $\Rightarrow$ $\boxed{\text{Converges}}$ by ACT

88. $\sum_{n=1}^{\infty}\frac{n!}{10^{4n}}$ $\boxed{\text{Diverges}}$ by Ratio Test because

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)!}{10^{4(n+1)}}}{\frac{n!}{10^{4n}}}\right| = \lim_{n\to\infty}\frac{\overset{(n+1)\cancel{n!}}{(n+1)!}}{\cancel{n!}}\cdot\frac{10^{4n}}{10^{4n+4}} = \lim_{n\to\infty}\frac{n+1}{10^4} = \infty > 1$$

89 $\sum_{n=1}^{\infty} e^{-2n} = \sum \frac{1}{e^{2n}}$ O.S. A.C. by R.T. $\Rightarrow$ O.S. $\boxed{\text{Converges}}$ by ACT

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{1}{e^{2n+2}}}{\frac{1}{e^{2n}}}\right| = \lim_{n\to\infty}\frac{e^{2n}}{e^{2n+2}} = \frac{1}{e^2} < 1$$

OR Geometric $\sum \frac{1}{e^{2n}} = \sum \frac{1}{(e^2)^n}$ $|r| = 1/e^2 < 1$ Convergent.

90. $\sum_{n=1}^{\infty} \frac{1+3n^3}{n^5} \approx \sum \frac{n^3}{n^5} = \sum \frac{1}{n^2}$ Convergent p-Series $p=2>1$

$$\lim_{n\to\infty}\frac{\frac{1+3n^3}{n^5}}{\frac{1}{n^2}} = \lim_{n\to\infty}\frac{n^2+3n^5}{n^5}\frac{\left(\frac{1}{n^5}\right)}{\left(\frac{1}{n^5}\right)} = \lim_{n\to\infty}\frac{\cancel{\frac{1}{n^3}}^{\,0}+3}{1} = 3$$ Finite, Non-zero

$\Rightarrow$ O.S. $\boxed{\text{Converges}}$ by LCT

91. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^7}$ Related Function ① Continuous for $x>1$

② Positive for $x>1$

③ Decreasing $f'(x) = \frac{-1}{(x(\ln x)^7)^2}\left[x \cdot 7(\ln x)^6 \cdot \frac{1}{x} + (\ln x)^7(1)\right]$

$= \frac{-(\ln x)^6\,[7+\ln x]}{(x(\ln x)^7)^2} < 0$ for $x>1$ o.K

$$\int_2^{\infty}\frac{1}{x(\ln x)^7}dx = \lim_{t\to\infty}\int_2^t \frac{1}{x(\ln x)^7}dx = \lim_{t\to\infty}\int_{\ln 2}^{\ln t}\frac{1}{u^7}du$$

$\boxed{\begin{array}{l} u=\ln x \\ du = \frac{1}{x}dx \end{array}}$ $\boxed{\begin{array}{l} x=2 \Rightarrow u=\ln 2 \\ x=t \Rightarrow u=\ln t \end{array}}$

$$= \lim_{t\to\infty} \left.\frac{-1}{6u^6}\right|_{\ln 2}^{\ln t}$$

$$= \lim_{t\to\infty} \cancel{\frac{-1}{6(\ln t)^6}}^{\,0} + \frac{1}{6(\ln 2)^6} = \frac{1}{6(\ln 2)^6}$$ Converges

$\Rightarrow$ O.S. $\boxed{\text{Converges}}$ by I.T.

92. $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan n}{1+n^2}$ Bound Terms $\displaystyle\frac{\arctan n}{1+n^2} \le \frac{\pi/2}{1+n^2} \le \frac{\pi/2}{n^2}$

and $\displaystyle\frac{\pi}{2}\sum \frac{1}{n^2}$ Constant Multiple of Convergent Series $p=2>1$ is Convergent

$\Rightarrow$ O.S. $\boxed{\text{Converges}}$ by CT

OR More work Integral Test also works

Related Function $f(x)=\dfrac{\arctan x}{1+x^2}$

① Continuous for all x
② Positive for $x>0$
③ Decreasing

$$f'(x)=\frac{\cancel{(1+x^2)}\frac{1}{\cancel{1+x^2}}-\arctan x(2x)}{(1+x^2)^2}$$

$$=\frac{1-2x\arctan x}{(1+x^2)^2 \; +} < 0$$

when $1-2x\arctan x<0$

$\frac{1}{2} < x\arctan x$ ok eventually

$x>\pi$ ?

$$\int_1^\infty \frac{\arctan x}{1+x^2}dx=\lim_{t\to\infty}\int_1^t \frac{\arctan x}{1+x^2}dx$$

u-sub?!

$$=\lim_{t\to\infty}\left.\frac{(\arctan x)^2}{2}\right|_1^t$$

$$=\lim_{t\to\infty}\frac{(\overset{\pi/2}{\cancel{\arctan t}})^2}{2}-\frac{(\overset{\pi/4}{\cancel{\arctan 1}})^2}{2}=\frac{1}{2}\left[\frac{\pi^2}{4}-\frac{\pi^2}{16}\right]=\boxed{\frac{3\pi^2}{32}}$$

Converges

$\Rightarrow$ O.S. $\boxed{\text{Converges}}$ by IT

93. $\displaystyle\sum_{n=1}^{\infty}\frac{n^7}{e^n}$

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{\frac{(n+1)^7}{e^{n+1}}}{\frac{n^7}{e^n}}\right|=\lim_{n\to\infty}\frac{(n+1)^7}{n^7}\cdot\frac{e^n}{e^{n+1}}$$

$$=\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^7\cdot\frac{1}{e}=\lim_{n\to\infty}\left(\frac{1+\cancel{\overset{0}{1/n}}}{1}\right)^7\cdot\frac{1}{e}=\frac{1}{e}<1$$

$\Rightarrow$ OS Converges Absolutely by RT

$\Rightarrow$ O.S. $\boxed{\text{Converges}}$ by ACT

94. $\sum_{n=1}^{\infty} \frac{n!\ln n}{n^2 \cdot 3^n}$

(*) L'H below

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)!\,\ln(n+1)}{(n+1)^2\,3^{n+1}}}{\frac{n!\,\ln n}{n^2\,3^n}}\right| = \lim_{n\to\infty} \frac{(n+1)!}{n!}\cdot\frac{n^2}{(n+1)^2}\cdot\frac{\ln(n+1)}{\ln n}\cdot\frac{3^n}{3^{n+1}}$$

$(n+1)n!$, $\left(\frac{n}{n+1}\right)^2$, $1$, $1/3$

$$=\lim_{n\to\infty}(n+1)\left(\frac{1}{1+1/n}\right)^2\cdot(1)\cdot\frac{1}{3} = \infty > 1 \quad \text{O.S.} \boxed{\text{Diverges}} \text{ by R.T.}$$

$$(*)\ \lim_{n\to\infty}\frac{\ln(n+1)}{\ln n} \overset{\infty/\infty}{=} \lim_{x\to\infty}\frac{\ln(x+1)}{\ln x} \overset{\infty/\infty}{=} \lim_{x\to\infty}\frac{\frac{1}{x+1}}{\frac{1}{x}} = \lim_{x\to\infty}\frac{x}{x+1} \overset{\infty/\infty}{\underset{L'H}{=}} \lim_{x\to\infty}\frac{1}{1} = 1$$

95. $\sum_{n=1}^{\infty}\frac{2n+5}{5n^3+3n} \approx \sum\frac{n}{n^3} = \sum\frac{1}{n^2}$ Convergent p-series $p=2>1$

$$\lim_{n\to\infty}\frac{\frac{2n+5}{5n^3+3n^2}}{1/n^2} = \lim_{n\to\infty}\frac{2n^3+5n^2\left(\frac{1}{n^3}\right)}{5n^3+3n^2\left(\frac{1}{n^3}\right)} = \lim_{n\to\infty}\frac{2+5/n^{\to 0}}{5+3/n^{\to 0}} = \frac{2}{5} \quad \text{Finite Non-Zero}$$

$\Rightarrow$ O.S also $\boxed{\text{Converges}}$ by LCT

96. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} \hookrightarrow \sum\frac{1}{n^2}$ Convergent p-series $p=2>1$

$\Rightarrow$ O.S. Converges by ACT

Very Quick Solution!

OR you can straight use AST (b/c they didn't ask A.C?)

① $b_n=\frac{1}{n^2}$

② $\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{n^2}=0$

③ $b_{n+1}\le b_n$ $\quad b_{n+1}=\frac{1}{(n+1)^2}\le\frac{1}{n^2}=b_n$ OR $f'(x)=\frac{-2}{x^3}<0$ for $x>0$. o.k

⊖ if $x>0$.

$\Rightarrow$ O.S. Converges by AST

97. $\sum_{n=1}^{\infty} \frac{1}{n+7} \approx \sum \frac{1}{n}$ Divergent Harmonic p-Series $p=1$

$$\lim_{n\to\infty} \frac{\frac{1}{n+7}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+7}\frac{(1/n)}{(1/n)} = \lim_{n\to\infty} \frac{1}{1+\cancel{7/n}^{\,0}} = 1 \quad \text{Finite, Non-Zero}$$

$\Rightarrow$ O.S. also $\boxed{\text{Diverges}}$ by LCT (CT Not Helpful Here)

98. $\sum_{n=1}^{\infty} \frac{n^2-1}{3n^2+1}$ Diverges by $n^{th}$ Term Divergence Test because

$$\lim_{n\to\infty} \frac{n^2-1}{3n^2+1}\frac{(1/n^2)}{(1/n^2)} = \lim_{n\to\infty} \frac{1-\cancel{1/n^2}^{\,0}}{3+\cancel{1/n^2}_{\,0}} = \frac{1}{3} \neq 0.$$

99. $\sum_{n=1}^{\infty} \frac{7}{25+n^2} \approx \sum \frac{1}{n^2}$ Convergent p-Series $p=2>1$

$$\lim_{n\to\infty} \frac{\frac{7}{25+n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{7n^2}{25+n^2}\frac{\left(\frac{1}{n^2}\right)}{\left(\frac{1}{n^2}\right)} = \lim_{n\to\infty} \frac{7}{\cancel{25/n^2}_{\,0}+1} = 7 \quad \text{Finite Non-Zero}$$

$\Rightarrow$ O.S. also $\boxed{\text{Converges}}$ by LCT

OR, use CT comparing $\sum \frac{1}{25+n^2}$ with $\sum \frac{1}{n^2}$ Conv. p-series $p=2>1$

Bound Terms

$\frac{1}{25+n^2} \le \frac{1}{n^2}$ and $\Rightarrow \sum \frac{1}{25+n^2}$ Converges by CT

$\Rightarrow 7\sum \frac{1}{25+n^2}$ Converges as a Constant Multiple of a Convergent Serie

100. $\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{2^{n+1}(n+1)!}{(n+1)^{n+1}}}{\frac{2^n n!}{n^n}}\right| = \lim_{n\to\infty} \frac{\cancel{2^{n+1}}^{\,2}}{2^n}\cdot\frac{\overbrace{(n+1)!}^{(n+1)\cancel{n!}}}{\cancel{n!}}\cdot\frac{n^n}{\underbrace{(n+1)^{n+1}}_{(n+1)^n\cancel{(n+1)}}} = \lim_{n\to\infty} 2\cdot\underbrace{\left(\frac{n}{n+1}\right)^n}_{1/e} = \frac{2}{e} < 1$$

$\Rightarrow$ O.S. A.C. by RT $\Rightarrow$ O.S. $\boxed{\text{Converges}}$ by ACT

101. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(2n-1)!}$

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)!}{(2(n+1)-1)!}}{\frac{n!}{(2n-1)!}}\right| = \lim_{n\to\infty} \frac{\overbrace{(n+1)!}^{(n+1)n!}}{\cancel{n!}}\cdot\frac{\cancel{(2n-1)!}}{\underbrace{(2n+1)!}_{(2n+1)(2n)\cancel{(2n-1)!}}} = \lim_{n\to\infty}\frac{n+1}{4n^2+2n}\frac{\left(\frac{1}{n^2}\right)}{\left(\frac{1}{n^2}\right)}$$

$$= \lim_{n\to\infty}\frac{\frac{1}{n}\to 0}{4+\frac{2}{n}\to 0} = 0 < 1 \quad \text{O.S. A.C. by RT.}$$

$\Rightarrow$ O.S. $\boxed{\text{Converges}}$ by ACT

102. $\displaystyle\sum_{n=1}^{\infty} 3+\frac{1}{3^n}$ Diverges by $n^{th}$ Term Divergence Test because

$$\lim_{n\to\infty} 3+\cancel{\frac{1}{3^n}} = 3 \neq 0.$$

103. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}}\right| = \lim_{n\to\infty}\frac{\overbrace{\cancel{(n+1)!}}^{\cancel{(n+1)n!}}}{\cancel{n!}}\cdot\frac{n^n}{\underbrace{(n+1)^{n+1}}_{(n+1)^n\cancel{(n+1)}}} = \lim_{n\to\infty}\frac{n^n}{(n+1)^n} = \frac{1}{e} < 1$$

$\frac{1}{e}$

$\Rightarrow$ O.S. A.C. by RT

$\Rightarrow$ O.S. $\boxed{\text{Converges}}$ by ACT

104. $\displaystyle\sum_{n=1}^{\infty} e^{1/n}$ Diverges by $n^{th}$ Divergence Test because

$$\lim_{n\to\infty} e^{1/n\to 0} = e^0 = 1 \neq 0$$

105. $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{((n+1)!)^2}{(2(n+1))!}}{\frac{(n!)^2}{(2n)!}}\right| = \lim_{n\to\infty}\frac{\overbrace{((n+1)!)^2}^{(n+1)^2\cancel{(n!)^2}}}{\cancel{(n!)^2}}\cdot\frac{\cancel{(2n)!}}{\underbrace{(2n+2)!}_{(2n+2)(2n+1)\cancel{(2n)!}}} = \lim_{n\to\infty}\frac{(n+1)^{\cancel{2}}}{2\cancel{(n+1)}(2n+1)} = \lim_{n\to\infty}\frac{n+1}{2(2n+1)}\frac{\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)}$$

$$= \lim_{n\to\infty}\frac{1+\frac{1}{n}\to 0}{2\left(2+\frac{1}{n}\right)\to 0} = \frac{1}{4} < 1 \Rightarrow \text{O.S. A.C. by } \boxed{\text{R.T.}}$$

$\Rightarrow$ O.S. $\boxed{\text{Converges}}$ by ACT

106. $\sum_{n=1}^{\infty} \frac{3}{n^3 7^n}$ Can use CT. Bound Terms $\frac{3}{n^3 7^n} \le \frac{3}{n^3}$ and $3\sum \frac{1}{n^3}$ Constant Multiple of Convergent is Convergent $\Rightarrow$ O.S. $\boxed{\text{Convergent}}$ by CT

OR R.T. $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{\cancel{3}}{(n+1)^3 7^{n+1}}}{\frac{\cancel{3}}{n^3 \cdot 7^n}} \right| = \lim_{n\to\infty} \frac{n^3}{(n+1)^3} \cdot \frac{7^n}{7^{n+1}} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^3 \cdot \frac{1}{7}$

$= \lim_{n\to\infty} \left(\frac{1}{1+\cancel{1/n}}\right)^3 \cdot \frac{1}{7} = \frac{1}{7} < 1$ O.S. A.C. by R.T.

$\Rightarrow$ O.S. $\boxed{\text{converges}}$ by ACT

107. $\sum_{n=1}^{\infty} \frac{2^n \cdot n^2}{(n+1)!}$

$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{2^{n+1}(n+1)^2}{(n+2)!}}{\frac{2^n n^2}{(n+1)!}} \right| = \lim_{n\to\infty} \frac{\cancel{2^{n+1}}}{\cancel{2^n}} \cdot \frac{(n+1)^2}{n^2} \cdot \frac{\cancel{(n+1)!}}{\underbrace{(n+2)!}_{(n+2)\cancel{(n+1)!}}} = \lim_{n\to\infty} 2 \cdot \left(\frac{1+\cancel{1/n}}{1}\right)^2 \cdot \frac{1}{n+2} = 0 < 1$

O.S. A.C. by R.T.

$\Rightarrow$ O.S. $\boxed{\text{Converges}}$ by ACT

108. $\sum_{n=1}^{\infty} \frac{e^{2n}\, n!}{9^n}$

$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{e^{2(n+1)}(n+1)!}{9^{n+1}}}{\frac{e^{2n}\, n!}{9^n}} \right| = \lim_{n\to\infty} \frac{\overset{e^2}{\cancel{e^{2n+2}}}}{\cancel{e^{2n}}} \cdot \frac{\overset{(n+1)n!}{(n+1)!}}{\cancel{n!}} \cdot \frac{\cancel{9^n}}{\underset{1/9}{\cancel{9^{n+1}}}} = \lim_{n\to\infty} e^2 \frac{(n+1)}{9} = \infty > 1$

O.S. $\boxed{\text{Diverges}}$ by R.T.

109. $\sum_{n=2}^{\infty} \frac{6}{n^6} + \frac{1}{(n+1)^6} = 6\overset{①}{\sum \frac{1}{n^6}} + \overset{②}{\sum \frac{1}{(n+1)^6}}$ $\boxed{\text{Converges}}$ as Sum of Two Convergent Series

① $6\sum \frac{1}{n^6}$ Constant Multiple of Convergent Series $p = 6 > 1$ is Convergent

② $\sum \frac{1}{(n+1)^6} \sim \sum \frac{1}{n^6}$ Convergent p-Series $p = 6 > 1$

Bound terms $\frac{1}{(n+1)^6} < \frac{1}{n^6}$ and $\nearrow$ $\Rightarrow$ O.S. Converges by CT

110. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)^n n!}{(2n)!}$

note: All n's replaced with n+1.

$(2n+2)^{n+1}$

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(2(n+1))^{n+1}(n+1)!}{(2(n+1))!}}{\frac{(2n)^n n!}{(2n)!}}\right| = \lim_{n\to\infty} \frac{\overbrace{(2n+2)^{n+1}}^{(2n+2)^n(2n+2)}}{(2n)^n}\cdot\frac{\overbrace{(n+1)!}^{(n+1)n!}}{n!}\cdot\frac{(2n)!}{\underbrace{(2n+2)!}_{(2n+2)(2n+1)(2n)!}}$$

$$=\lim_{n\to\infty}\left(\frac{2n+2}{2n}\right)^n\cdot\frac{n+1}{2n+1} = \lim_{n\to\infty}\left(\frac{2(n+1)}{2n}\right)^n\cdot\frac{1+1/n}{2+1/n} \to \frac{1}{2}$$

$$=\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n\cdot\frac{1}{2} = \frac{e}{2} > 1 \quad \text{O.S. } \boxed{\text{Diverges}} \text{ by R.T.}$$

111. $\displaystyle\sum_{n=1}^{\infty} \frac{4^n (n!)^3}{(2n)!\, n^n}$

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{4^{n+1}((n+1)!)^3}{(2(n+1))!\,(n+1)^{n+1}}}{\frac{4^n(n!)^3}{(2n)!\,n^n}}\right| = \lim_{n\to\infty}\frac{4^{n+1}}{4^n}\cdot\frac{[\overbrace{(n+1)!}^{(n+1)^3(n!)^3}]^3}{(n!)^3}\cdot\frac{(2n)!}{\underbrace{(2n+2)!}_{(2n+2)(2n+1)(2n)!}}\cdot\frac{n^n}{\underbrace{(n+1)^{n+1}}_{(n+1)^n(n+1)}}$$

$$=\lim_{n\to\infty}\frac{4\cdot(n+1)^3}{\underbrace{(2n+2)}_{2(n+1)}(2n+1)(n+1)}\cdot\frac{n^n}{(n+1)^n} = \lim_{n\to\infty}\frac{4}{2}\left(\frac{n+1}{2n+1}\right)\frac{1}{e} = \lim_{n\to\infty}\frac{2}{e}\cdot\left(\frac{1+1/n}{2+1/n}\right) = \frac{1}{e} < 1$$

O.S. A.C. by R.T.

$\Rightarrow$ O.S. $\boxed{\text{Converges}}$ by ACT

Even More Series

112. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{5n} \xrightarrow{A.S.} \sum \frac{1}{5n} = \frac{1}{5}\sum \frac{1}{n}$ Constant Multiple of Divergent Harmonic p-Series $p=1$ is Divergent

AST ① $b_n = \frac{1}{5n} > 0$

② $\lim_{n\to\infty} \frac{1}{5n} = 0$

③ $b_{n+1} \le b_n$ OR $f'(x) = \frac{-5}{(5x)^2} < 0$

$b_{n+1} = \frac{1}{5(n+1)} = \frac{1}{5n+5} \le \frac{1}{5n} = b_n$

O.S. Converges by AST

$\boxed{C.C.}$

113. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{2^n}$

$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{\cancel{(-1)^{n+2}}(n+1)}{2^{n+1}}}{\frac{\cancel{(-1)^{n+1}}\, n}{2^n}}\right| = \lim_{n\to\infty} \left(\frac{n+1}{n}\right) \cdot \frac{2^n}{2^{n+1}} = \lim_{n\to\infty} \left(\frac{1+\frac{1}{n}}{1}\right) \cdot \frac{1}{2} = \frac{1}{2} < 1$

O.S. $\boxed{A.C.}$ by RT.

114. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{7n-3} \xrightarrow{A.S.} \sum_{n=1}^{\infty} \frac{1}{7n-3} \approx \sum \frac{1}{n}$ Divergent Harmonic Series $p=1$

$\lim_{n\to\infty} \frac{\frac{1}{7n-3}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{7n-3} \frac{\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{1}{7-\frac{3}{n}} = \frac{1}{7}$ Finite Non-zero

$\Rightarrow$ A.S. also Diverges by LCT

1) $b_n = \frac{1}{7n-3} > 0$

② $\lim_{n\to\infty} \frac{1}{7n-3} = 0$

③ $b_{n+1} \le b_n$

$\frac{1}{\underbrace{7(n+1)-3}_{7n+4}} \le \frac{1}{7n-3}$ ✓

OR $f(x) = \frac{1}{7x-3}$ has $f'(x) = \frac{-7}{(7x-3)^2} < 0 \Rightarrow f \downarrow$.

O.S. Converges by AST

$\boxed{C.C.}$

115. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{10n+1}$

$$\lim_{n\to\infty} \frac{n}{10n+1}\left(\frac{1/n}{1/n}\right) = \lim_{n\to\infty} \frac{1}{10+\underbrace{1/n}_{\to 0}} = \frac{1}{10} \neq 0$$

$\Rightarrow \lim_{n\to\infty} (-1)^{n+1}\frac{n}{10n+1} \neq 0$ (DNE actually) $\Rightarrow$ O.S. Diverges by $n^{th}$ Term Divergence Test

116. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2+1} \xrightarrow{A.S.}$ FIRST $\sum_{n=1}^{\infty} \frac{n}{n^2+1} \approx \sum \frac{1}{n}$ Diverges Harmonic p-Series p=1

$$\lim_{n\to\infty} \frac{\frac{n}{n^2+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2}{n^2+1}\left(\frac{1/n^2}{1/n^2}\right) = \lim_{n\to\infty} \frac{1}{1+\underbrace{1/n^2}_{\to 0}} = 1 \text{ Finite Non-zero}$$

$\Rightarrow$ A.S. Diverges by LCT

LAST
AST on O.S

① $b_n = \frac{n}{n^2+1} > 0$

② $\lim_{n\to\infty} \frac{n}{n^2+1}\left(\frac{1/n^2}{1/n^2}\right) = \lim_{n\to\infty} \frac{\overset{\to 0}{1/n}}{1+\underbrace{1/n^2}_{\to 0}} = 0$

③ $b_{n+1} \le b_n$ b/c $f(x) = \frac{x}{x^2+1}$

has $f'(x) = \frac{(x^2+1)(1) - x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} < 0$ when $x>1$ o.k.

O.S. Converges by AST

$\Rightarrow \boxed{C.C.}$

117. $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n} \xrightarrow{A.S} \sum_{n=1}^{\infty} \frac{\ln n}{n}$ Will Diverge by IT or CT

① IT $f(x) = \frac{\ln x}{x}$ Continuous $x>0$; Positive $x>1$; Decreasing $f'(x) = \frac{x(1/x) - \ln x(1)}{x^2} = \frac{1-\ln x}{x^2} < 0$ when $1 < \ln x$, $e < x$ o.k.

$$\int_1^{\infty} \frac{\ln x}{x}dx = \lim_{t\to\infty}\int_1^t \frac{\ln x}{x}dx = \lim_{t\to\infty} \left.\frac{(\ln x)^2}{2}\right|_1^t$$

$$= \lim_{t\to\infty} \frac{\overset{\to\infty}{(\ln t)^2}}{2} - \frac{\overset{\to 0}{(\ln 1)^2}}{2} = \infty \text{ Diverges} \Rightarrow \text{A.S. Diverges by IT}$$

OR ② Bound Terms $\frac{\ln n}{n} \ge \frac{1}{n}$ (for $n \ge 3$) and $\sum \frac{1}{n}$ Divergent p-Seri[es] Harmonic p=1

$\Rightarrow$ A.S. Diverges by CT

AST
① $b_n = \frac{\ln n}{n} > 0$

② $\lim_{n\to\infty} \frac{\ln n}{n} \overset{\infty/\infty}{=} \lim_{x\to\infty} \frac{\ln x}{x} \overset{\infty/\infty}{=} \lim_{x\to\infty} \frac{\overset{\to 0}{1/x}}{1} = 0$

③ $b_{n+1} \le b_n$
$f(x) = \frac{\ln x}{x}$

$f'(x) = \frac{x(1/x) - \ln x(1)}{x^2}$ (as shown above right)

$= \frac{1-\ln x}{x^2} < 0$ when $x \ge e$ O.K

O.S. Converges by AST

$\boxed{C.C.}$

118. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n!}{2^{n^2}}$

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{\cancel{(-1)^{n+2}}(n+1)!}{2^{(n+1)^2}}}{\frac{\cancel{(-1)^{n+1}}\, n!}{2^{n^2}}} \right| = \lim_{n\to\infty} \frac{\overbrace{(n+1)!}^{(n+1)\cancel{n!}}}{\cancel{n!}} \cdot \frac{2^{n^2}}{\underbrace{2^{n^2+2n+1}}_{2^{n^2}\cdot 2^{2n+1}}} = \lim_{n\to\infty} \frac{n+1}{2\cdot 2^{2n}} \overset{\infty/\infty}{=} \lim_{x\to\infty} \frac{x+1}{2\cdot 2^{2x}} \overset{\infty/\infty}{}$$

$$= \lim_{x\to\infty} \frac{1}{\underbrace{2\cdot 2^{2x}\ln 2(2)}_{\infty}} = 0 < 1$$

O.S. $\boxed{\text{A.C.}}$ by R.T.

119. $\sum_{n=2}^{\infty} \frac{n(-3)^{2n+1}}{10^n}$

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{(n+1)(-3)^{2(n+1)+1}}{10^{n+1}}}{\frac{n(-3)^{2n+1}}{10^n}} \right| = \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{\overset{3^2}{\cancel{3^{2n+3}}}}{\cancel{3^{2n+1}}} \cdot \underbrace{\frac{\cancel{10^n}}{\cancel{10^{n+1}}}}_{1/10} = \lim_{n\to\infty} \overset{1}{\cancel{1+\tfrac{1}{n}}}^{\,0} \cdot \frac{9}{10} = \frac{9}{10} < 1$$

O.S. $\boxed{\text{A.C.}}$ by RT

120. $\sum_{n=1}^{\infty} \frac{7^n}{n^n}$

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{7^{n+1}}{(n+1)^{n+1}}}{\frac{7^n}{n^n}} \right| = \lim_{n\to\infty} \frac{\overset{7}{\cancel{7^{n+1}}}}{\cancel{7^n}} \cdot \frac{n^n}{\underbrace{(n+1)^{n+1}}_{(n+1)^n(n+1)}} = \lim_{n\to\infty} \underbrace{\cancel{\frac{7}{n+1}}}_{\infty}{}^{0} \cdot \underbrace{\cancel{\frac{n^n}{(n+1)^n}}}_{1/e} = 0 < 1$$

O.S. $\boxed{\text{A.C.}}$ by RT

121. $\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}$

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{e^{2(n+1)}}{(n+1)^{n+1}}}{\frac{e^{2n}}{n^n}} \right| = \lim_{n\to\infty} \frac{\overset{e^2}{\cancel{e^{2n+2}}}}{\cancel{e^{2n}}} \cdot \frac{n^n}{\underbrace{(n+1)^{n+1}}_{(n+1)^n(n+1)}} = \lim_{n\to\infty} \underbrace{\cancel{\frac{e^2}{n+1}}}_{\infty}{}^{0} \cdot \underbrace{\frac{n^n}{(n+1)^n}}_{\cancel{1/e}} = 0 < 1$$

O.S. $\boxed{\text{A.C.}}$ by RT

122. $\displaystyle\sum_{n=1}^{\infty} \frac{(-4)^{2n+1}}{n \cdot 10^n}$

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{(-4)^{2(n+1)+1}}{(n+1)\,10^{n+1}}}{\frac{(-4)^{2n+1}}{n \cdot 10^n}} \right| = \lim_{n\to\infty} \frac{\cancel{4^{2n+3}}^{\,4^2}}{\cancel{4^{2n+1}}} \cdot \frac{n}{n+1} \cdot \frac{\cancel{10^n}}{\cancel{10^{n+1}}_{\,1/10}} = \lim_{n\to\infty} \frac{16}{10} \cdot \left(\frac{1}{1+\underbrace{1/n}_{0}}\right) = \frac{16}{10} > 1$$

O.S. $\boxed{\text{Diverges}}$ by RT

123. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\arctan n}{n^7+n} \xrightarrow{A.S.} \sum \frac{\arctan n}{n^7+n}$

$\boxed{CT}$ Bound Terms

$$\frac{\arctan n}{n^7+n} \le \frac{\pi/2}{n^7+n} \le \frac{\pi/2}{n^7}$$

and $\frac{\pi}{2} \sum \frac{1}{n^7}$ Constant Multiple of a Convergent p-Series $p=7>1$ is Convergent

OR

$\boxed{LCT}$ $\approx \sum \frac{1}{n^7}$ Conv. p-Series $p=7>1$

$$\lim_{n\to\infty} \frac{\frac{\arctan n}{n^7+n}}{1/n^7} = \lim_{n\to\infty} \arctan n \cdot \frac{n^7}{n^7+n} \frac{\left(\frac{1}{n^7}\right)}{\left(\frac{1}{n^7}\right)}$$

$$= \lim_{n\to\infty} \underbrace{\arctan n}_{\pi/2} \left(\frac{1}{\underbrace{1+1/n^6}_{1}}\right) = \frac{\pi}{2}$$ Finite Non-zero

$\Rightarrow$ A.S. also Converges by LCT

$\Rightarrow$ $\boxed{A.C.}$

$\Rightarrow$ A.S. Converges by CT $\Rightarrow$ $\boxed{A.C.}$ Done!

(Side Note: $\Rightarrow$ O.S. Converges by ACT, not needed)

124. $\displaystyle\sum_{n=1}^{\infty} \frac{(n+2)!}{3^n (n!)^2}$

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{(n+3)!}{3^{n+1}[(n+1)!]^2}}{\frac{(n+2)!}{3^n(n!)^2}} \right| = \lim_{n\to\infty} \frac{\cancel{(n+3)!}^{\,(n+3)\cancel{(n+2)!}}}{\cancel{(n+2)!}} \cdot \frac{\cancel{3^n}}{\cancel{3^{n+1}}_{\,1/3}} \cdot \frac{\cancel{(n!)^2}}{[(n+1)!]^2_{\,(n+1)^2(n!)^2}} = \lim_{n\to\infty} \frac{n+3}{3(n+1)^2} = \lim_{n\to\infty} \frac{n+3}{3(n^2+2n+1)}$$

$$= \lim_{n\to\infty} \frac{\underbrace{1/n}_{0} + \underbrace{3/n^2}_{0}}{3(1+\underbrace{2/n}_{0}+\underbrace{1/n^2}_{0})} = \frac{0}{3} = 0 < 1$$

$\Rightarrow$ O.S. $\boxed{A.C.}$ by RT

125. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (3n)!\, n^2}{8^n (n!)^2\, n^n}$

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\dfrac{\cancel{(-1)^{n+1}} (3(n+1))!\,(n+1)^2}{8^{n+1}((n+1)!)^2 (n+1)^{n+1}}}{\dfrac{\cancel{(-1)^n} (3n)!\, n^2}{8^n (n!)^2 n^n}} \right|$$

$$= \lim_{n\to\infty} \frac{\overbrace{(3n+3)!}^{(3n+3)(3n+2)(3n+1)\cancel{(3n)!}}}{\cancel{(3n)!}} \cdot \frac{n^2}{(n+1)^2} \cdot \underbrace{\frac{\cancel{8^n}}{8^{n+1}}}_{1/8} \cdot \frac{\cancel{(n!)^2}}{\underbrace{((n+1)!)^2}_{(n+1)^2\cancel{(n!)^2}}} \cdot \frac{n^n}{\underbrace{(n+1)^{n+1}}_{(n+1)^n (n+1)}}$$

$3(n+1)$

$$= \lim_{n\to\infty} (3n+3)(3n+2)(3n+1) \cdot \left(\frac{n(\frac{1}{n})}{n+1(\frac{1}{n})}\right)^2 \cdot \frac{1}{8} \frac{1}{(n+1)^2} \cdot \underbrace{\frac{n^n}{(n+1)^n}}_{1/e} \cdot \cancel{\frac{1}{n+1}}$$

$$= \lim_{n\to\infty} 3\left(\frac{3n+2}{n+1}\right)\left(\frac{3n+1}{n+1}\right)\left(\frac{1}{1+\frac{1}{n}}\right)^2 \cdot \frac{1}{8} \cdot \frac{1}{e}$$

$$= \lim_{n\to\infty} 3\left(\frac{3+\cancel{2/n}}{1+\cancel{1/n}}\right) \cdot \left(\frac{3+\cancel{1/n}}{1+\cancel{1/n}}\right) \cdot \frac{1}{8e}$$

$$= \frac{27}{8e} > 1 \qquad \text{O.S. Diverges by R.T}$$

126. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \ln n \cdot \pi^n (2n)!}{n^n\, 4^n\, n!}$

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\dfrac{\cancel{(-1)^{n+1}} \ln(n+1)\, \pi^{n+1} [2(n+1)]!}{(n+1)^{n+1}\, 4^{n+1} (n+1)!}}{\dfrac{\cancel{(-1)^n} \ln n\, \pi^n (2n)!}{n^n\, 4^n\, n!}} \right|$$

$$= \lim_{n\to\infty} \underbrace{\frac{\ln(n+1)}{\ln n}}_{\substack{\text{See}(x)\ 1 \\ \text{L'H}}} \cdot \frac{\pi^{n+1}}{\pi^n} \cdot \frac{\overbrace{(2n+2)!}^{(2n+2)(2n+1)(2n)!}}{(2n)!} \cdot \frac{n^n}{\underbrace{(n+1)^{n+1}}_{(n+1)^n(n+1)}} \cdot \underbrace{\frac{4^n}{4^{n+1}}}_{1/4} \cdot \frac{n!}{\underbrace{(n+1)!}_{(n+1)n!}}$$

$$= \lim_{n\to\infty} (1) \cdot \pi \frac{\overbrace{(2n+2)}^{2(n+1)}(2n+1)}{\cancel{(n+1)}(n+1)} \cdot \underbrace{\frac{n^n}{(n+1)^n}}_{\to 1/e} \cdot \frac{1}{4}$$

$$= \lim_{n\to\infty} \frac{2\pi}{4e} \left(\frac{2n+1}{n+1}\right)\frac{1/n}{1/n}$$

$$= \lim_{n\to\infty} \frac{2\pi}{4e} \cdot \left[\frac{2 + \cancel{1/n}^{\,0}}{1 + \cancel{1/n}^{\,0}}\right] = \frac{4\pi}{4e} = \frac{\pi}{e} > 1$$

O.S. $\boxed{\text{Diverges}}$ by R.T.