Answer Key for midterm 2 sample 1

1. [30 Points] Compute the following integral, or else show that it diverges.

(a) $\displaystyle\int_1^3 \frac{x+3}{x^3+3x}\,dx = \int_1^3 \left(\frac{1}{x} - \frac{x}{x^2+3} + \frac{1}{x^2+3}\right) dx$ (see the computation below)

$\displaystyle = \ln|x| - \frac{\ln|x^2+3|}{2} + \frac{1}{\sqrt{3}}\arctan\left(\frac{x}{\sqrt{3}}\right) + C$

Partial Fractions Decomposition:

$$\frac{x+3}{x(x^2+3)} = \frac{A}{x} + \frac{Bx+C}{x^2+3}$$

Clearing the denominator yields:

$x+3 = A(x^2+3) + (Bx+C)x$
$x+3 = (A+B)x^2 + Cx + 3A$
so that $A+B=0,\ C=1$ and $3A=3$
Solve for $A=1,\ C=1$ and $B=-1$

(b) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{36+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{36+x^2}\,dx + \int_0^{\infty} \frac{1}{36+x^2}\,dx$

$$= \lim_{s\to-\infty}\int_s^0 \frac{1}{36+x^2}\,dx + \lim_{t\to\infty}\int_0^t \frac{1}{36+x^2}\,dx$$

$$= \lim_{s\to-\infty} \frac{1}{6}\arctan\left(\frac{x}{6}\right)\Big|_s^0 + \lim_{t\to\infty}\frac{1}{6}\arctan\left(\frac{x}{6}\right)\Big|_0^t$$

$$= \lim_{s\to-\infty}\frac{1}{6}\arctan(0) - \frac{1}{6}\arctan\left(\frac{s}{6}\right) + \lim_{t\to\infty}\frac{1}{6}\arctan\left(\frac{t}{6}\right) - \frac{1}{6}\arctan(0)$$

$$= \frac{1}{6}\left(0 - \left(-\frac{\pi}{2}\right) + \left(\frac{\pi}{2} - 0\right)\right) = \frac{1}{6}\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = \boxed{\frac{\pi}{6}}$$

(c) $\displaystyle\int_1^{\infty} \frac{e^{\frac{1}{x}}}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{e^{\frac{1}{x}}}{x^2}\,dx = \lim_{t\to\infty}\int_1^{1/t} (-e^u)\,du = \lim_{t\to\infty}\left[-e^u\right]_1^{1/t}$

$\displaystyle = \lim_{t\to\infty} -e^{1/t} - \left(-e^1\right) = \boxed{e-1}$ because $\displaystyle\lim_{t\to\infty} e^{\frac{1}{t}} = e^0 = 1$

Where we have used the substitution $\boxed{\begin{array}{l} u = \frac{1}{x} \\ du = -\frac{1}{x^2}dx \\ -du = \frac{1}{x^2}dx \end{array}}$ $\boxed{\begin{array}{l} x = t \Rightarrow u = \frac{1}{t} \\ x = 1 \Rightarrow u = 1 \end{array}}$

2. [10 Points] Determine **and state** whether the following sequence **converges** or **diverges**. If it converges, compute its limit. Justify your answer. Do **not** just put down a number.

$$\left\{\left(\frac{n}{n+5}\right)^{2n+1}\right\}_{n=1}^{\infty}$$

Switch to the variable x and the related function $f(x) = \left(\frac{x}{x+5}\right)^{2x+1}$ in order to apply L'H Rule:

$$\lim_{n\to\infty}\left(\frac{n}{n+5}\right)^{2n+1^{(1^\infty)}} \quad \lim_{x\to\infty}\left(\frac{x}{x+5}\right)^{2x+1^{(1^\infty)}}$$

$$= \lim_{x\to\infty} e^{\ln\left(\frac{x}{x+5}\right)^{2x+1}} = e^{\lim\limits_{x\to\infty}\ln\left[\left(\frac{x}{x+5}\right)^{2x+1}\right]} = e^{\lim\limits_{x\to\infty}(2x+1)\ln\left(\frac{x}{x+5}\right)^{(\infty\cdot 0)}}$$

$$= e^{\lim\limits_{x\to\infty}\frac{\ln\left(\frac{x}{x+5}\right)^{\frac{0}{0}}}{\frac{1}{2x+1}}} \overset{\text{L'H}}{=} e^{\lim\limits_{x\to\infty}\frac{\left(\frac{x+5}{x}\right)\left(\frac{(x+5)(1)-x(1)}{(x+5)^2}\right)}{-\frac{2}{(2x+1)^2}}}$$

$$= e^{\lim\limits_{x\to\infty}\frac{\left(\frac{x+5}{x}\right)\left(\frac{5}{(x+5)^2}\right)}{-\frac{2}{(2x+1()^2}}} = e^{\lim\limits_{x\to\infty}\left(\frac{x+5}{x}\right)\left(\frac{5}{(x+5)^2}\right)\left(-\frac{(2x+1)^2}{2}\right)}$$

$$= e^{\lim\limits_{x\to\infty}\left(\frac{-5(4x^2+4x+1)}{2(x^2+5x)}\right)} = e^{\lim\limits_{x\to\infty}\left(\frac{-5\left(4+\frac{4}{x}+\frac{1}{x^2}\right)}{2\left(1+\frac{5}{x}\right)}\right)} = e^{\frac{-20}{2}} = e^{-10} = \boxed{\frac{1}{e^{10}}} \; \boxed{\text{Converges}}$$

3. [10 Points] Find the **sum** of the following series (which does converge).

$$\sum_{n=1}^{\infty}(-1)^n\frac{5^{2n+1}}{2^{5n-1}} = -\frac{5^3}{2^4}+\frac{5^5}{2^9}-\frac{5^7}{2^{14}}+\dots$$

Here we have a geometric series with $a = -\frac{125}{16}$ and $r = -\frac{5^2}{2^5} = -\frac{25}{32}$. Note, it does converge since $|r| = \left|-\frac{25}{32}\right| = \frac{25}{32} < 1$.

As a result, the sum is given by SUM$=\frac{a}{1-r} = \frac{-\frac{125}{16}}{1-\left(-\frac{25}{32}\right)} = \frac{-\frac{125}{16}}{\frac{57}{32}} = -\frac{125}{16}\cdot\frac{32}{57} = \boxed{-\frac{250}{57}}$

4. [10 Points] Use the **Integral Test** to **determine** and **state** whether the series $\sum\limits_{n=1}^{\infty}\frac{\ln n}{n^3}$ converges or diverges. Justify all of your work.

Consider the related function $f(x) = \dfrac{\ln x}{x^3}$ with

1. $f(x)$ continuous for all $x > 0$
2. $f(x)$ positive for $x > 1$
3. $f(x)$ decreasing because

$$f'(x) = \frac{x^3\left(\frac{1}{x}\right) - \ln x(3x^2)}{x^6} = \frac{x^2 - 3x^2\ln x}{x^6} = \frac{1 - 3\ln x}{x^4} < 0 \text{ when } x > e^{\frac{1}{3}}.$$

Check the improper integral

$$\int_1^\infty \frac{\ln x}{x^3}\,dx = \lim_{t\to\infty} \int_1^t \ln x\left(x^{-3}\right)\,dx = \lim_{t\to\infty} -\frac{\ln x}{2x^2}\bigg|_1^t + \frac{1}{2}\int_1^t x^{-3}\,dx$$

$$= \lim_{t\to\infty} -\frac{\ln x}{2x^2}\bigg|_1^t - \frac{1}{4x^2}\bigg|_1^t = \lim_{t\to\infty} -\frac{\ln t}{2t^2}\,\frac{\infty}{\infty} - \left(-\frac{\ln 1}{2}\right) - \left(\frac{1}{4t^2} - \frac{1}{4}\right)$$

$$\overset{\text{L'H}}{=} -\frac{\frac{1}{t}}{4t} + 0 - \frac{1}{4t^2} + \frac{1}{4} = -\frac{1}{4t^2} + 0 - \frac{1}{4t^2} + \frac{1}{4} = \boxed{\frac{1}{4}}$$

The improper integral converges, and therefore the original series **Converges** by the Integral Test (IT).

IBP:

$u = \ln x$	$dv = x^{-3}dx$
$du = \frac{1}{x}dx$	$v = -\frac{1}{2x^2}$

5. [15 Points] Determine whether each of the following series **converges** or **diverges**. Name any convergence test(s) you use, and justify all of your work.

(a) $\displaystyle\sum_{n=1}^{\infty} \arctan\left(\frac{\sqrt{3}\,n^3 + 1}{n^3 + n}\right)$

Diverges by n^{th} term Divergence Test

since $$\lim_{n\to\infty} \arctan\left(\frac{\sqrt{3}\,n^3 + 1}{n^3 + n}\right) = \arctan\left(\lim_{n\to\infty} \frac{(\sqrt{3}\,n^3 + 1)}{(n^3 + n)}\,\frac{\left(\frac{1}{n^3}\right)}{\left(\frac{1}{n^3}\right)}\right)$$

$$= \arctan\left(\lim_{n\to\infty} \frac{\sqrt{3} + \frac{1}{n^3}}{1 + \frac{1}{n^2}}\right) = \arctan(\sqrt{3}) = \frac{\pi}{3} \neq 0$$

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\arctan(\sqrt{3}\,n^3 + 1)}{n^3 + n}$

First examine the absolute series $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan(\sqrt{3}\, n^3+1)}{n^3+n}$

Next bound the terms

$$\frac{\arctan(\sqrt{3}\, n^3+1)}{n^3+n} < \frac{\frac{\pi}{2}}{n^3+n} < \frac{\frac{\pi}{2}}{n^3}$$

and

$\displaystyle\frac{\pi}{2}\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a constant multiple of a convergent p-series with $p = 3 > 1$ and therefore convergent.

Finally, the absolute series is Convergent by CT, and therefore the orginal series is $\boxed{\text{Convergent}}$ by ACT.

Or more simply, A.S. CONV by CT $\Rightarrow$ O.S. CONV by ACT.

6. [25 Points] In each case determine whether the given series is **absolutely convergent**, **conditionally convergent**, or **diverges**. Name any convergence test(s) you use, and justify all of your work.

(a) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^3+7n}{n^9+\sqrt{n}}$

First examine the absolute series

$$\sum_{n=1}^{\infty} \frac{n^3+7n}{n^9+\sqrt{n}}$$

Note that $\displaystyle\sum_{n=1}^{\infty} \frac{n^3+7n}{n^9+\sqrt{n}} \approx \sum_{n=1}^{\infty} \frac{n^3}{n^9} = \sum_{n=1}^{\infty} \frac{1}{n^6}$

which is a convergent p-series with $p = 6 > 1$. Next,

Check: $\displaystyle\lim_{n\to\infty} \frac{\frac{n^3+7n}{n^9+\sqrt{n}}}{\frac{1}{n^6}} = \lim_{n\to\infty} \frac{n^9+7n^7}{n^9+\sqrt{n}} \frac{\left(\frac{1}{n^9}\right)}{\left(\frac{1}{n^9}\right)} = \lim_{n\to\infty} \frac{1+\frac{7}{n^2}}{1+\frac{1}{n^{\frac{17}{2}}}} = 1$ which is finite and non-zero.

Therefore, these two series share the same behavior, and the absolute series $\displaystyle\sum_{n=1}^{\infty} \frac{n^3+7n}{n^9+\sqrt{n}}$ is also Convergent, by Limit Comparison Test (LCT). Or more simply, A.S. CONV by LCT. Finally, we have $\boxed{\text{Absolute Convergence (A.C.)}}$

(Not needed here but **Note**: This implies that the Original Series is Convergent by ACT.)

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \cdot 15^n}{(n!)^2}$

Try Ratio Test:

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}\cdot 15^{n+1}}{((n+1)!)^2}}{\frac{(-1)^n\cdot 15^n}{(n!)^2}}\right|$$

$$= \lim_{n\to\infty}\frac{15^{n+1}}{15^n}\cdot\frac{(n!)^2}{((n+1)!)^2}$$

$$= \lim_{n\to\infty} 15\cdot\frac{(n!)^2}{(n+1)^2(n!)^2}$$

$$= \lim_{n\to\infty}\frac{15}{(n+1)^2}$$

$= 0$ (which is less than 1)

The original series is $\boxed{\text{Absolutely Convergent by the Ratio Test}}$.

(c) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\,\frac{1}{\sqrt{n}+7}$

First, we show the absolute series is divergent. Note that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}+7}\approx\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ which is a divergent p-series with $p=\dfrac{1}{2}<1$. Next,

Check: $\displaystyle\lim_{n\to\infty}\frac{\frac{1}{\sqrt{n}+7}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n}+7}\frac{\left(\frac{1}{\sqrt{n}}\right)}{\left(\frac{1}{\sqrt{n}}\right)} = \lim_{n\to\infty}\frac{1}{1+\frac{7}{\sqrt{n}}} = 1$ which is finite and non-zero.

Therefore, these two series share the same behavior. Since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ is the divergent p-Series $\left(p=\dfrac{1}{2}<1\right)$, then the absolute series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}+7}$ is also divergent by Limit Comparison Test. Or more simply, A.S. DIV by LCT. As a result, we have no chance for Absolute Convergence.

Secondly, we are left to examine the original alternating series with the Alternating Series Test.

- $b_n = \dfrac{1}{\sqrt{n}+7} > 0$

- $\displaystyle\lim_{n\to\infty}\frac{1}{\sqrt{\ n}+7} = 0$

- $\dfrac{1}{b_{n+1}} < \dfrac{1}{b_n}$ because $b_{n+1} = \dfrac{1}{\sqrt{n+1}+7} < \dfrac{1}{\sqrt{n}+7} = b_n$

OR to show terms decreasing, could also show that for $f(x) = \dfrac{1}{\sqrt{x}+7}$,

we have $f'(x) = -\left(\dfrac{1}{2\sqrt{x}(\sqrt{x}+7)^2}\right) < 0$.

Therefore, the original series converges by the Alternating Series Test. (Or simply O.S. CONV by

AST) Finally, we can conclude the original series is $\boxed{\text{Conditionally Convergent (C.C.)}}$