Math 121 Midterm Exam #2, sample 2

Answer Key

1. Compute the following integral.

(a) $\displaystyle\int \frac{x+7}{x^3+7x}\,dx = \int \frac{x+7}{x(x^2+7)}\,dx$

$\displaystyle= \int \left(\frac{1}{x} + \frac{-x+1}{x^2+7}\right)dx = \int \left(\frac{1}{x} - \frac{x}{x^2+7} + \frac{1}{x^2+7}\right)dx$

$\displaystyle= \boxed{\ln|x| - \frac{\ln|x^2+7|}{2} + \frac{1}{\sqrt{7}}\arctan\left(\frac{x}{\sqrt{7}}\right) + C}$

Partial Fractions Decomposition:

$\displaystyle\frac{x+7}{x(x^2+7)} = \frac{A}{x} + \frac{Bx+C}{x^2+7}$

Clearing the denominator yields:

$x+7 = A(x^2+7) + (Bx+C)x$

$x+7 = (A+B)x^2 + Cx + 7A$

so that $A+B=0,\ C=1$ and $7A=7$

Solve for $A=1,\ C=1$ and $B=-1$

(b) $\displaystyle\int_2^\infty \frac{1}{x\ \ln x}\,dx = \lim_{t\to\infty}\int_2^t \frac{1}{x\ \ln x}\,dx$

$\displaystyle= \lim_{t\to\infty}\int_{\ln 2}^{\ln t}\frac{1}{u}\,du = \lim_{t\to\infty}\left[\ln|\ln|x||\right]_{\ln 2}^{\ln t}$

$\displaystyle= \lim_{t\to\infty}(\ln(\ln t) - \ln(\ln 2)) = \infty - \ln(\ln 2) = \boxed{\infty \text{ (diverges)}}$

Substitute $\boxed{\begin{array}{l} u = \ln x \\ du = \frac{1}{x}dx \end{array}}$ $\boxed{\begin{array}{l} x = t \Rightarrow u = \ln t \\ x = \frac{1}{2} \Rightarrow u = \ln\left(\frac{1}{2}\right) \end{array}}$

(c) $\displaystyle\int_8^\infty \frac{1}{x^2-10x+28}\,dx = \lim_{t\to\infty}\int_8^t \frac{1}{(x-5)^2+3}\,dx$ complete the square

Substitute $\boxed{\begin{array}{l} u = x-5 \\ du = dx \end{array}}$ $\boxed{\begin{array}{l} x = 8 \Rightarrow u = 3 \\ x = t \Rightarrow u = t-5 \end{array}}$

$\displaystyle= \lim_{t\to\infty}\int_3^{t-5}\frac{1}{u^2+3}\,du = \lim_{t\to\infty}\frac{1}{\sqrt{3}}\arctan\left(\frac{u}{\sqrt{3}}\right)\Bigg|_3^{t-5} = \lim_{t\to\infty}\frac{1}{\sqrt{3}}\left(\arctan\left(\frac{t-5}{\sqrt{3}}\right) - \arctan\left(\frac{3}{\sqrt{3}}\right)\right)$

$\displaystyle= \lim_{t\to\infty}\frac{1}{\sqrt{3}}\left(\arctan\left(\frac{t-5}{\sqrt{3}}\right) - \arctan\left(\sqrt{3}\right)\right)$

$$= \frac{1}{\sqrt{3}}\left(\frac{\pi}{2} - \frac{\pi}{3}\right) = \frac{1}{\sqrt{3}}\left(\frac{3\pi}{6} - \frac{2\pi}{6}\right) = \frac{1}{\sqrt{3}}\left(\frac{\pi}{6}\right) = \boxed{\frac{\pi}{6\sqrt{3}}}$$

using the formula $\displaystyle\int \frac{1}{x^2 + a^2}\, dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$

2.

(a) Determine **and state** whether the following *sequence* **converges** or **diverges**. If it converges, compute its limit. Justify your answer. Do **not** just put down a number.

Solution:

Switch to the variable x and consider the function

$$f(x) = \frac{\sqrt{2x^4 + 5x^3 + 7}}{1 + 5x^2}.$$

The limit of the sequence is equal to limit of this function, as x goes to infinity. This is

$$\begin{aligned}
\lim_{x\to\infty} \frac{\sqrt{2x^4 + 5x^3 + 7}}{1 + 5x^2} &= \lim_{x\to\infty} \frac{\sqrt{2x^4 + 5x^3 + 7}}{1 + 5x^2} \cdot \frac{1/x^2}{1/x^2} \\
&= \lim_{x\to\infty} \frac{\sqrt{2 + 5/x + 7/x^4}}{1/x^2 + 5} \\
&= \frac{\sqrt{2 + 0 + 0}}{0 + 5} \\
&= \frac{\sqrt{2}}{5}
\end{aligned}$$

So the limit of the sequence is $\frac{\sqrt{2}}{5}$.

(b) Determine **and state** whether the following *series* **converges** or **diverges**. Justify your answer.

$$\sum_{n=1}^{\infty} \frac{\sqrt{2n^4 + 5n^3 + 7}}{1 + 5n^2}$$

Solution:

We found in part (a) that the limit of the terms of this sequence is $\frac{\sqrt{2}}{5}$. This is not zero, so the series **diverges** by NTDT.

3. Find the **sum** of the following series (which does converge).

$$\sum_{n=1}^{\infty}(-1)^n \frac{5^{n+1}}{3^{2n-1}}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n\,5^{n+1}}{3^{2n-1}} = -\frac{5^2}{3}+\frac{5^3}{3^3}-\frac{5^4}{3^5}+\dots$$

Here we have a geometric series with $a=-\frac{25}{3}$ and $r=-\frac{5}{3^2}=-\frac{5}{9}$. Note, it does converge since $|r|=\left|-\frac{5}{9}\right|=\frac{5}{9}<1$.

As a result, the sum is given by SUM$=\frac{a}{1-r}=\frac{-\frac{25}{3}}{1-\left(-\frac{5}{9}\right)}=\frac{-\frac{25}{3}}{\frac{14}{9}}=-\frac{25}{3}\cdot\frac{9}{14}=\boxed{-\frac{75}{14}}$

4. Use the **Integral Test** to **determine** and **state** whether the series $\sum_{n=1}^{\infty}\frac{n}{e^{3n}}$ converges or diverges. Justify all of your work.

Consider the related function $f(x)=\frac{x}{e^{3x}}$ with

1. $f(x)$ continuous for all x
2. $f(x)$ positive for $x>0$
3. $f(x)$ decreasing because $f'(x)=\frac{e^{3x}(1)-x\left(3e^{3x}\right)}{\left(e^{3x}\right)^2}=\frac{1-3x}{e^{3x}}<0$ when $x>\frac{1}{3}$.

Check the improper integral

$$\int_1^{\infty}\frac{x}{e^{3x}}\,dx=\lim_{t\to\infty}\int_1^t xe^{-3x}\,dx=\lim_{t\to\infty}-\frac{1}{3}xe^{-3x}\Big|_1^t+\frac{1}{3}\int_1^t e^{-3x}\,dx$$

$$=\lim_{t\to\infty}-\frac{1}{3}xe^{-3x}\Big|_1^t-\frac{1}{9}e^{-3x}\Big|_1^t$$

$$=\lim_{t\to\infty}-\frac{1}{3}\left(\frac{t}{e^{3t}}\right)^{\frac{\infty}{\infty}}-\left(-\frac{1}{3e^3}\right)-\frac{1}{9e^{3t}}-\left(-\frac{1}{9e^3}\right)$$

$$\stackrel{\text{L'H}}{=}\lim_{t\to\infty}-\left(\frac{1}{9e^{3t}}\right)+\frac{1}{3e^3}+\frac{1}{9e^3}=\frac{3}{9e^3}+\frac{1}{9e^3}=\frac{4}{9e^3}$$

The improper integral converges, and therefore the original series $\boxed{\text{Converges}}$ by the Integral Test (IT).

Or more simply O.S. CONV by IT. IBP:

$u=x$	$dv=e^{-3x}dx$
$du=dx$	$v=-\frac{1}{3}e^{-3x}$

5. Determine whether each of the following series **converges** or **diverges**. Name any convergence test(s) you use, and justify all of your work.

(a) $\displaystyle\sum_{n=1}^{\infty} n\sin\left(\frac{1}{n}\right)$ $\boxed{\text{Diverges by } n^{th} \text{ term Divergence Test}}$

since $\displaystyle\lim_{n\to\infty} n\sin\left(\frac{1}{n}\right) \overset{\infty\cdot 0}{=} \lim_{x\to\infty} x\sin\left(\frac{1}{x}\right)$

$$= \lim_{x\to\infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} \overset{\text{L}'H}{=} \lim_{x\to\infty} \frac{\cos\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x\to\infty} \cos\left(\frac{1}{x}\right) = 1 \neq 0$$

(b) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{n^3} + \frac{\sin^2(3n)}{3^n}\right) = \sum_{n=1}^{\infty} \frac{3}{n^3} + \sum_{n=1}^{\infty} \frac{\sin^2(3n)}{3^n}$

First note that $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n^3} = 3\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a Constant multiple of a Convergent p-Series with $p = 3 > 1$ and is therefore convergent, **and** $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2(3n)}{3^n}$ is convergent by CT because the terms are bounded $\displaystyle\frac{\sin^2(3n)}{3^n} < \frac{1}{3^n}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$ is convergent geometric series with $|r| = \frac{1}{3} < 1$.

Finally, the original series is $\boxed{\text{Convergent}}$ because it is the sum of two convergent series.

6. In each case determine whether the given series is **absolutely convergent**, **conditionally convergent**, or **diverges**. Name any convergence test(s) you use, and justify all of your work.

(a) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^2+7}{n^7+2}$

First examine the absolute series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+7}{n^7+2}$

Note that $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+7}{n^7+2} \approx \sum_{n=1}^{\infty} \frac{n^2}{n^7} = \sum_{n=1}^{\infty} \frac{1}{n^5}$ which is a convergent p-series with $p = 5 > 1$. Next,

Check: $\displaystyle\lim_{n\to\infty} \frac{\frac{n^2+7}{n^7+2}}{\frac{1}{n^5}} = \lim_{n\to\infty} \frac{n^7+7n^5}{n^7+2} = \lim_{n\to\infty} \frac{1+\frac{7}{n^2}}{1+\frac{2}{n^7}} = 1$ which is finite and non-zero ($0 < 1 < \infty$).

Therefore, these two series share the same behavior, and the absolute series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+7}{n^7+2}$ is also Convergent, by Limit Comparison Test (LCT). (Or more simply A.S CONV by LCT) Finally, we have $\boxed{\text{Absolute Convergence (A.C.)}}$.

(Not needed here but **Note**: this implies that the Original Series is Convergent by ACT.)

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n (n!)^2}{(2n)!}$

Try Ratio Test:

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-3)^{n+1}((n+1)!)^2}{(2(n+1))!} \cdot \frac{(2n)!}{(-3)^n (n!)^2}\right| = \lim_{n\to\infty} \left|\frac{(-3)^{n+1}}{(-3)^n} \frac{((n+1)!)^2}{(n!)^2} \cdot \frac{(2n)!}{(2n+2)!}\right|$$

$$= \lim_{n\to\infty} \left|(-3)(n+1)^2 \frac{1}{(2n+1)(2n+2)}\right| = 3 \lim_{n\to\infty} \frac{(n+1)^2}{(2n+1)\cdot 2(n+1)} = \frac{3}{2} \lim_{n\to\infty} \frac{n+1}{2n+1} = \frac{3}{4}$$

(The last step is justified by, for example, l'Hôpital's rule.)

This limit L is less than 1, so the original series is $\boxed{\text{Absolutely Convergent (A.C.) by the Ratio Test}}$.
(Or more simply O.S. AC by RT)

(c) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n}+4}$

First, we show the absolute series is divergent. Note that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+4} \approx \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which is a divergent p-series with $p = \dfrac{1}{2} < 1$. Next,

Check: $\displaystyle\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}+4}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}+4} = \lim_{n\to\infty} \frac{1}{1+\frac{4}{\sqrt{n}}} = 1$ which is finite and non-zero. Therefore, these two series share the same behavior. Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is the divergent p-Series $\left(p = \dfrac{1}{2} < 1\right)$, then $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+4}$ is also divergent by Limit Comparison Test. As a result, we have no chance for Absolute Convergence.

(Or more simply, A.S. DIV by LCT.)

Secondly, we are left to examine the original alternating series with the Alternating Series Test.

- $b_n = \dfrac{1}{\sqrt{n}+4} > 0$

- $\displaystyle\lim_{n\to\infty} \frac{1}{\sqrt{n}+4} = 0$

- $\dfrac{1}{b_{n+1}} < \dfrac{1}{b_n}$ because $b_{n+1} = \dfrac{1}{\sqrt{n+1}+4} < \dfrac{1}{\sqrt{n}+4} = b_n$

OR to show terms decreasing, could also show that for $f(x) = \frac{1}{\sqrt{x}+4}$,

we have $f'(x) = -\left(\frac{1}{2\sqrt{x}(\sqrt{x}+4)^2}\right) < 0$.

Therefore, the original series converges by the Alternating Series Test. (Or more simply, O.S. CONV by AST.) Finally, we can conclude the original series is $\boxed{\text{Conditionally Convergent (C.C.)}}$