Math 121 Midterm Exam #1 February 21, 2014

Answer Key

1. [10 Points]

(a) Let $y = \arcsin x$. Use implicit differentiation to **PROVE** that $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-x^2}}$.

If $y = \arcsin x$, then $\sin y = x$. Implicitly differentiate both sides,

$\dfrac{d}{dx}(\sin y) = \dfrac{d}{dx}(x)$.

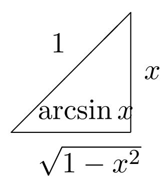
Then $\cos y \dfrac{dy}{dx} = 1$.

Solve for $\dfrac{dy}{dx} = \dfrac{1}{\cos y} = \dfrac{1}{\sqrt{1-\sin^2 y}} = \dfrac{1}{\sqrt{1-x^2}}$ Here we used the trig. identity to finish

OR finish

$\dfrac{dy}{dx} = \dfrac{1}{\cos y} = \dfrac{1}{\cos(\arcsin x)} = \dfrac{1}{\sqrt{1-x^2}}$

using the trig. from the triangle



(b) From part (a) we now know that $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$. You may use this fact to **PROVE** that

$$\int \frac{1}{\sqrt{9-x^2}}\,dx = \arcsin\left(\frac{x}{3}\right) + C \qquad \longleftarrow \textbf{Prove this.}$$

$$\int \frac{1}{\sqrt{9-x^2}}\,dx = \int \frac{1}{\sqrt{9\left(1-\frac{x^2}{9}\right)}}\,dx = \int \frac{1}{\sqrt{9}\sqrt{1-\left(\frac{x}{3}\right)^2}}\,dx$$

$$= \frac{1}{3}\int \frac{1}{\sqrt{1-\left(\frac{x}{3}\right)^2}}\,dx = \int \frac{1}{\sqrt{1-u^2}}\,du = \arcsin u = \arcsin\left(\frac{x}{3}\right) + C$$

Standard u substitution to simplify:

$u = \dfrac{x}{3}$
$du = \dfrac{1}{3}\,dx$

Note: **OR** you can also do a trig. substitution here.

2. [30 Points] Evaluate each of the following **limits**. Please justify your answers. Be clear if the limit equals a value, $+\infty$ or $-\infty$, or Does Not Exist.

(a) $$\lim_{x\to 0} \frac{5xe^x - \arctan(5x)}{\sinh x + \ln(1-x)} \left(\frac{0}{0}\right) \overset{\text{L'H}}{=} \lim_{x\to 0} \frac{5xe^x + 5e^x - \dfrac{5}{1+(5x)^2}}{\cosh x + \dfrac{-1}{1-x}} \left(\frac{0}{0}\right)$$

$$\overset{\text{L'H}}{=} \lim_{x\to 0} \frac{5xe^x + 5e^x + 5e^x + \dfrac{5(50x)}{(1+25x^2)^2}}{\sinh x + \dfrac{-1}{(1-x)^2}} = \frac{10}{-1} = \boxed{-10}$$

(b) $$\lim_{x\to\infty} \left(e^{\frac{1}{x}} - \frac{4}{x}\right)^x \overset{1^\infty}{=} \lim_{x\to\infty} e^{\ln\left(\left(e^{\frac{1}{x}} - \frac{4}{x}\right)^x\right)} = e^{\lim_{x\to\infty} \ln\left(\left(e^{\frac{1}{x}} - \frac{4}{x}\right)^x\right)} = e^{\lim_{x\to\infty} x \ln\left(e^{\frac{1}{x}} - \frac{4}{x}\right)}$$

$$\overset{\infty\cdot 0}{=} e^{\lim_{x\to\infty} \frac{\ln\left(e^{\frac{1}{x}} - \frac{4}{x}\right)}{\frac{1}{x}}} \overset{\left(\frac{0}{0}\right)\text{L'H}}{=} e^{\lim_{x\to\infty} \frac{\left(\frac{1}{e^{\frac{1}{x}} - \frac{4}{x}}\right)\cdot\left(e^{\frac{1}{x}}\left(-\frac{1}{x^2}\right) + \frac{4}{x^2}\right)}{-\frac{1}{x^2}}}$$

$$= e^{\lim_{x\to\infty} \left(\frac{1}{e^{\frac{1}{x}} - \frac{4}{x}}\right)\cdot\left(e^{\frac{1}{x}}\left(-\frac{1}{x^2}\right) + \frac{4}{x^2}\right)(-x^2)} = e^{\lim_{x\to\infty} \left(\frac{1}{e^{\frac{1}{x}} - \frac{4}{x}}\right)\cdot\left(e^{\frac{1}{x}}(1) - 4\right)} = \boxed{e^{-3}}$$

(c) $$\lim_{x\to\infty} (\ln x)^{\frac{3}{x}} \overset{\infty^0}{=} \lim_{x\to\infty} e^{\ln\left((\ln x)^{\frac{3}{x}}\right)} = e^{\lim_{x\to\infty} \ln\left((\ln x)^{\frac{3}{x}}\right)} = e^{\lim_{x\to\infty} \left(\frac{3}{x}\right)\ln(\ln x)}$$

$$e^{\lim_{x\to\infty} \left(\frac{3\ln(\ln x)}{x}\right)} \overset{\frac{\infty}{\infty}}{\underset{\text{L'H}}{=}} e^{\lim_{x\to\infty} \left(\frac{\left(\frac{3}{\ln x}\right)\left(\frac{1}{x}\right)}{1}\right)} = e^{\lim_{x\to\infty} \left(\frac{3}{x\ln x}\right)} = e^0 = \boxed{1}$$

3. [30 Points] Compute the following **definite integral**. Please simplify your answer.

(a) $$\int_0^{\ln 7} x\sinh x\, dx == x\cosh x\Big|_0^{\ln 7} - \int_0^{\ln 7} \cosh x\, dx = x\cosh x\Big|_0^{\ln 7} - \sinh x\Big|_0^{\ln 7}$$

$$= \ln 7\cosh(\ln 7) - 0 - \sinh(\ln 7) + \sinh 0 = \ln 7\cosh(\ln 7) - \sinh(\ln 7) + 0$$

$$= \ln 7\left(\frac{e^{\ln 7} + e^{-\ln 7}}{2}\right) - \left(\frac{e^{\ln 7} - e^{-\ln 7}}{2}\right) = \ln 7\left(\frac{7+\frac{1}{7}}{2}\right) - \left(\frac{7-\frac{1}{7}}{2}\right)$$

$$= \ln 7 \left(\frac{\frac{50}{7}}{2}\right) - \left(\frac{\frac{48}{7}}{2}\right) = \boxed{\ln 7 \left(\frac{25}{7}\right) - \left(\frac{24}{7}\right)}$$

I.B.P. $\boxed{\begin{array}{ll} u = x & dv = \sinh x dx \\ du = dx & v = \cosh x \end{array}}$

(b) $\displaystyle\int_3^{3\sqrt{3}} \frac{1}{\sqrt{36 - x^2}} + \frac{1}{9 + x^2}\ dx = \arcsin\left(\frac{x}{6}\right)\Bigg|_3^{3\sqrt{3}} + \frac{1}{3}\arctan\left(\frac{x}{3}\right)\Bigg|_3^{3\sqrt{3}}$

$$= \arcsin\left(\frac{3\sqrt{3}}{6}\right) - \arcsin\left(\frac{3}{6}\right) + \frac{1}{3}\arctan\left(\frac{3\sqrt{3}}{3}\right) - \frac{1}{3}\arctan\left(\frac{3}{3}\right)$$

$$= \arcsin\left(\frac{\sqrt{3}}{2}\right) - \arcsin\left(\frac{1}{2}\right) + \frac{1}{3}\arctan\left(\sqrt{3}\right) - \frac{1}{3}\arctan(1)$$

$$= \frac{\pi}{3} - \frac{\pi}{6} + \frac{1}{3}\left(\frac{\pi}{3}\right) - \frac{1}{3}\left(\frac{\pi}{4}\right) = \frac{\pi}{3} - \frac{\pi}{6} + \frac{\pi}{9} - \frac{\pi}{12} = \frac{12\pi}{36} - \frac{6\pi}{36} + \frac{4\pi}{36} - \frac{3\pi}{36} = \boxed{\frac{7\pi}{36}}$$

(c) $\displaystyle\int_1^e \frac{1}{x[1 + (\ln x)^2]}\ dx = \arctan(\ln x)\Big|_1^e = \arctan(\ln e) - \arctan(\ln 1)$

$$= \arctan(1) - \arctan(0) = \frac{\pi}{4} - 0 = \boxed{\frac{\pi}{4}}$$

OR you can use a standard u-substitution, being careful to *change* the limits of integration or *mark them* as x-limits. $\boxed{\begin{array}{l} u = \ln x \\ du = \frac{1}{x}\ dx \end{array}}$ $\boxed{\begin{array}{l} x = 1 \Rightarrow u = \ln(1) = 0 \\ x = e \Rightarrow u = \ln(e) = 1 \end{array}}$

$$\int_1^e \frac{1}{x[1 + (\ln x)^2]}\ dx = \int_0^1 \frac{1}{1 + u^2}\ du = \arctan u\Big|_0^1 = \arctan(1) - \arctan(0) = \frac{\pi}{4} - 0 = \boxed{\frac{\pi}{4}}$$

OR

$$\int_1^e \frac{1}{x[1 + (\ln x)^2]}\ dx = \int_{x=1}^{x=e} \frac{1}{1 + u^2}\ du = \arctan u\Big|_{x=1}^{x=e} = \arctan(\ln x)\Big|_1^e = \ldots$$

4. [30 Points] Compute the following **indefinite integral**.

(a) $\displaystyle\int x \arcsin x\ dx = \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{x^2}{\sqrt{1 - x^2}}\ dx = \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{\sin^2\theta}{\sqrt{1 - \sin^2\theta}} \cdot \cos\theta\ d\theta$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{\sin^2\theta}{\sqrt{\cos^2\theta}} \cdot \cos\theta\ d\theta = \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{\sin^2\theta}{\cos\theta} \cdot \cos\theta\ d\theta = \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \sin^2\theta d\theta$$

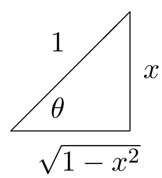
$$= \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{1-\cos(2\theta)}{2} d\theta = \frac{x^2}{2}\arcsin x - \frac{1}{4}\int 1-\cos(2\theta) d\theta$$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{4}\left[\theta - \frac{1}{2}\sin(2\theta)\right] + C = \frac{x^2}{2}\arcsin x - \frac{1}{4}\theta + \frac{1}{8}\sin(2\theta) + C$$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{4}\theta + \frac{1}{8}2\sin\theta\cos\theta + C = \boxed{\frac{x^2}{2}\arcsin x - \frac{1}{4}\arcsin x + \frac{1}{4}x\sqrt{1-x^2} + C}$$

$$\boxed{\begin{array}{ll} u = \arcsin x & dv = x dx \\ du = \frac{1}{\sqrt{1-x^2}} dx & v = \frac{x^2}{2} \end{array}}$$

Trig. Substitute $\boxed{\begin{array}{l} x = \sin\theta \\ dx = \cos\theta d\theta \end{array}}$



(b) $\displaystyle\int \frac{e^x}{(e^{2x}+4)^{\frac{7}{2}}}\ dx = \int \frac{1}{(u^2+4)^{\frac{7}{2}}}\ du = \int \frac{1}{(4\tan^2\theta+4)^{\frac{7}{2}}} \cdot\ 2\sec^2\theta\ d\theta$

$$= \int \frac{1}{(4\sec^2\theta)^{\frac{7}{2}}} \cdot\ 2\sec^2\theta\ d\theta = \int \frac{1}{(\sqrt{4\sec^2\theta})^7} \cdot\ 2\sec^2\theta\ d\theta = \int \frac{1}{(2\sec\theta)^7} \cdot\ 2\sec^2\theta\ d\theta$$

$$= \frac{1}{2^6}\int \frac{\sec^2\theta}{\sec^7\theta}\ d\theta = \frac{1}{2^6}\int \frac{1}{\sec^5\theta}\ d\theta = \frac{1}{64}\int \cos^5\theta\ d\theta$$

$$= \frac{1}{64}\int \cos^4\theta\cos\theta\ d\theta\ = \frac{1}{64}\int (\cos^2\theta)^2\cos\theta\ d\theta\ = \frac{1}{64}\int (1-\sin^2\theta)^2\cos\theta\ d\theta$$

$$= \frac{1}{64}\int (1-w^2)^2\ dw = \frac{1}{64}\int 1 - 2w^2 + w^4\ dw$$

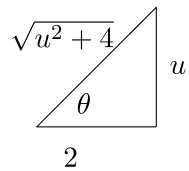
$$= \frac{1}{64}\left(w - \frac{2w^3}{3} + \frac{w^5}{5}\right) + C = \frac{1}{64}\left(\sin\theta - \frac{2\sin^3\theta}{3} + \frac{\sin^5\theta}{5}\right) + C$$

$$= \boxed{\frac{1}{64}\left(\frac{u}{\sqrt{u^2+4}} - \frac{2u^3}{3(u^2+4)^{\frac{3}{2}}} + \frac{u^5}{5(u^2+4)^{\frac{5}{2}}} + C\right)}$$

$$= \boxed{\frac{1}{64}\left(\frac{e^x}{\sqrt{e^{2x}+4}} - \frac{2e^{3x}}{3(e^{2x}+4)^{\frac{3}{2}}} + \frac{e^{5x}}{5(e^{2x}+4)^{\frac{5}{2}}}\right) + C}$$

Standard u substitution to simplify at the start: $\boxed{\begin{array}{l} u = e^x \\ du = e^x dx \end{array}}$

Trig. Substitute $\boxed{\begin{array}{l} u = 2\tan\theta \\ \\ du = 2\sec^2\theta d\theta \end{array}}$



Standard w substitution for *odd* trig. integral $\int \cos^5\theta\ d\theta$ technique:

$$\boxed{\begin{array}{l} w = \sin\theta \\ \\ dw = \cos\theta\ d\theta \end{array}}$$

(c) $\int \ln(x^2+1)\ dx = x\ln(x^2+1) - 2\int \frac{x^2}{x^2+1}\ dx$

$= x\ln(x^2+1) - 2\int \frac{x^2+1-1}{x^2+1}\ dx = x\ln(x^2+1) - 2\left(\int \frac{x^2+1}{x^2+1}\ dx - \int \frac{1}{x^2+1}\ dx\right)$

$= x\ln(x^2+1) - 2\left(\int 1\ dx - \int \frac{1}{x^2+1}\ dx\right)$

$= x\ln(x^2+1) - 2\left(x - \arctan x\right) + C = \boxed{x\ln(x^2+1) - 2x + 2\arctan x + C}$

I.B.P. $\boxed{\begin{array}{ll} u = \ln(x^2+1) & dv = dx \\ \\ du = \frac{2x}{x^2+1}dx & v = x \end{array}}$

NOTE: **OR** you can use a trig substitution to finish $\int \frac{x^2}{x^2+1}\ dx$.

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OPTIONAL BONUS

Do not attempt these unless you are completely done with the rest of the exam.

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OPTIONAL BONUS #1

Compute the following **indefinite integral**.

1. $\int e^{\sqrt{1+\sqrt{x}}}\ dx = 4\int w(w^2-1)e^w\ dw = 4\int (w^3-w)e^w\ dw = 4\left(\int w^3e^w\ dw - \int we^w\ dw\right)$

$= 4\left((*) - (**)\right) = 4\left(\left(w^3e^w - 3w^2e^w + 6we^w - 6e^w\right) - \left(we^w - e^w\right)\right) + C$

$= 4\left(w^3e^w - 3w^2e^w + 6we^w - 6e^w - we^w + e^w\right) + C = 4\left(w^3e^w - 3w^2e^w + 5we^w - 5e^w\right) + C$

$= 4\left(\left(\sqrt{1+\sqrt{x}}\right)^3 e^{\sqrt{1+\sqrt{x}}} - 3\left(\sqrt{1+\sqrt{x}}\right)^2 e^{\sqrt{1+\sqrt{x}}} + 5\left(\sqrt{1+\sqrt{x}}\right)e^{\sqrt{1+\sqrt{x}}} - 5e^{\sqrt{1+\sqrt{x}}}\right) + C$

$$= 4e^{\sqrt{1+\sqrt{x}}}\left[\left(\sqrt{1+\sqrt{x}}\right)^3 - 3\left(\sqrt{1+\sqrt{x}}\right)^2 + 5\left(\sqrt{1+\sqrt{x}}\right) - 5\right] + C$$

$$= 4e^{\sqrt{1+\sqrt{x}}}\left[(1+\sqrt{x})\sqrt{1+\sqrt{x}} - 3(1+\sqrt{x}) + 5\left(\sqrt{1+\sqrt{x}}\right) - 5\right] + C$$

$$= 4e^{\sqrt{1+\sqrt{x}}}\left[\sqrt{1+\sqrt{x}} + \sqrt{x}\left(\sqrt{1+\sqrt{x}}\right) - 3 - 3\sqrt{x} + 5\left(\sqrt{1+\sqrt{x}}\right) - 5\right] + C$$

$$= \boxed{4e^{\sqrt{1+\sqrt{x}}}\left[6\sqrt{1+\sqrt{x}} + \sqrt{x}\left(\sqrt{1+\sqrt{x}}\right) - 8 - 3\sqrt{x}\right] + C}$$

Here
$$\boxed{\begin{aligned} w &= \sqrt{1+\sqrt{x}} \Longrightarrow w^2 = 1+\sqrt{x} \Longrightarrow w^2 - 1 = \sqrt{x} \\ dw &= \left(\frac{1}{2\sqrt{1+\sqrt{x}}}\right)\left(\frac{1}{2\sqrt{x}}\right)dx \\ 4\left(\sqrt{1+\sqrt{x}}\right)\sqrt{x}dw &= dx \\ 4w(w^2-1)dw &= dx \end{aligned}}$$

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(*)Aside: $\int w^3e^w\ dw = w^3e^w - 3\int w^2e^w\ dw = w^3e^w - 3\left(w^2e^w - 2\int we^w\ dw\right)$

$$= w^3e^w - 3w^2e^w + 6\int we^w\ dw = w^3e^w - 3w^2e^w + 6\left(we^w - \int e^w\ dw\right)$$

$$= w^3e^w - 3w^2e^w + 6\int we^w\ dw = w^3e^w - 3w^2e^w + 6we^w - 6e^w + C$$

First I.B.P.

$u = w^3$	$dv = e^w dw$
$du = 3w^2dw$	$v = e^w$

Second I.B.P.

$u = w^2$	$dv = e^w dw$
$du = 2wdw$	$v = e^w$

Third I.B.P.

$u = w$	$dv = e^w dw$
$du = dw$	$v = e^w$

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(**)Aside: $\int we^w\ dw = we^w - \int e^w\ dw = we^w - e^w + C$

I.B.P.

$u = w$	$dv = e^w dw$
$du = dw$	$v = e^w$

OPTIONAL BONUS #2

Compute the following **indefinite integral**.

2. $\displaystyle\int \frac{\ln(x-1)}{\sqrt{x}}\ dx = \int \frac{\ln((\sqrt{x})^2 - 1)}{\sqrt{x}}\ dx = 2\int \ln(u^2-1)\ dx = 2\int \ln((u-1)(u+1))\ du$

$$= 2\int \ln(u-1)+\ln(u+1)\ du = 2\left((u-1)\ln(u-1)-(u-1)+(u+1)\ln(u+1)-(u+1)\right)+C$$

$$= 2\left((\sqrt{x}-1)\ln(\sqrt{x}-1)-(\sqrt{x}-1)+(\sqrt{x}+1)\ln(\sqrt{x}+1)-(\sqrt{x}+1)\right)+C$$

$$= 2\left((\sqrt{x}-1)\ln(\sqrt{x}-1)-\sqrt{x}+1+(\sqrt{x}+1)\ln(\sqrt{x}+1)-\sqrt{x}-1\right)+C$$

$$= \boxed{2\left((\sqrt{x}-1)\ln(\sqrt{x}-1)-2\sqrt{x}+(\sqrt{x}+1)\ln(\sqrt{x}+1)\right)+C}$$

Here $\boxed{\begin{aligned} u &= \sqrt{x} \\ du &= \tfrac{1}{2\sqrt{x}}dx \\ 2du &= \tfrac{1}{\sqrt{x}}dx \end{aligned}}$

OR recognize $\displaystyle\int \frac{\ln(x-1)}{\sqrt{x}}\ dx = \int \frac{\ln((\sqrt{x})^2-1)}{\sqrt{x}}\ dx = \int \frac{\ln((\sqrt{x}-1)(\sqrt{x}+1))}{\sqrt{x}}\ dx$

$$= \int \frac{\ln(\sqrt{x}-1)+\ln(\sqrt{x}+1)}{\sqrt{x}}\ dx = \int \frac{\ln(\sqrt{x}-1)}{\sqrt{x}}\ dx + \int \frac{\ln(\sqrt{x}+1)}{\sqrt{x}}\ dx$$

$=\ldots$ do subsitution on both pieces, and then I.B.P.

$$= 2\left((\sqrt{x}-1)\ln(\sqrt{x}-1)-(\sqrt{x}-1)+(\sqrt{x}+1)\ln(\sqrt{x}+1)-(\sqrt{x}+1)\right)+C$$