Math 121 Midterm Exam #1 October 5, 2016

Answer Key

1. [10 Points] Use implicit differentiation to **PROVE** that $\frac{d}{dx}\arcsin(3x) = \frac{3}{\sqrt{1-9x^2}}$.

Let $y = \arcsin(3x)$. Then $\sin y = 3x$. Implicitly differentiate both sides,

$\frac{d}{dx}(\sin y) = \frac{d}{dx}(3x)$.

Then $\cos y \frac{dy}{dx} = 3$.

Solve for $\frac{dy}{dx} = \frac{3}{\cos y} = \frac{3}{\sqrt{1-\sin^2 y}} = \frac{3}{\sqrt{1-(3x)^2}}$

Here we used the identity $\cos^2 x + \sin^2 x = 1$, and the fact that $\cos x = +\sqrt{1-\sin^2 x}$ because $\cos x \geq 0$ for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.

2. [30 Points] Evaluate each of the following **limits**. Please justify your answers. Be clear if the limit equals a value, $+\infty$ or $-\infty$, or Does Not Exist.

(a) $$\lim_{x\to 0^+} (1-3\sin x)^{\frac{1}{x}} \overset{1^\infty}{=} e^{\lim_{x\to 0^+} \ln\left[(1-3\sin x)^{\frac{1}{x}}\right]}$$

$$= e^{\lim_{x\to 0^+} \frac{\ln(1-3\sin x)}{x}} \overset{\left(\frac{0}{0}\right)\text{L'H}}{=} e^{\lim_{x\to 0^+} \frac{\frac{1}{1-3\sin x}(-3\cos x)}{1}} = \boxed{e^{-3}}$$

(b) $$\lim_{x\to 0} \frac{\arcsin x + \cos(3x) - e^x}{\arctan(3x) + x^2 - \sinh(3x)} \overset{\left(\frac{0}{0}\right)\text{L'H}}{=} \lim_{x\to 0} \frac{\frac{1}{\sqrt{1-x^2}} - 3\sin(3x) - e^x}{\frac{3}{1+9x^2} + 2x - 3\cosh(3x)}$$

$$\overset{\left(\frac{0}{0}\right)\text{L'H}}{=} \lim_{x\to 0} \frac{-\frac{1}{2(1-x^2)^{\frac{3}{2}}}(-2x) - 9\cos(3x) - e^x}{-\frac{3}{(1+9x^2)^2}(18x) + 2 - 9\sinh(3x)}$$

$$= \lim_{x\to 0} \frac{\frac{x}{(1-x^2)^{\frac{3}{2}}} - 9\cos(3x) - e^x}{-\frac{54x}{(1+9x^2)^2} + 2 - 9\sinh(3x)} = \frac{-9-1}{2} = \frac{-10}{2} = \boxed{-5}$$

(c) $$\lim_{x\to\infty} \left(1 - \arcsin\left(\frac{5}{x^4}\right)\right)^{3x^4} \overset{1^\infty}{=} e^{\lim_{x\to\infty} \ln\left[\left(1-\arcsin\left(\frac{5}{x^4}\right)\right)^{3x^4}\right]}$$

$$= e^{\lim_{x\to\infty} 3x^4 \ln\left(1-\arcsin\left(\frac{5}{x^4}\right)\right)} = e^{\lim_{x\to\infty} \frac{3\ln\left(1-\arcsin\left(\frac{5}{x^4}\right)\right)}{\frac{1}{x^4}}} \quad \left(\frac{0}{0}\right)$$

$$\overset{\text{L'H}}{=} e^{\lim_{x\to\infty} \frac{\left(\frac{3}{1-\arcsin\left(\frac{5}{x^4}\right)}\right)\left(-\frac{1}{\sqrt{1-\left(\frac{5}{x^4}\right)^2}}\right)\left(-\frac{20}{x^5}\right)}{-\frac{4}{x^5}}}$$

$$= e^{\lim_{x\to\infty} \left(\frac{3}{1-\arcsin\left(\frac{5}{x^4}\right)}\right)\left(-\frac{1}{\sqrt{1-\left(\frac{5}{x^4}\right)^2}}\right)(5)} = \boxed{e^{-15}}$$

3. [45 Points] Compute the following **definite integral**. Please simplify your answer.

(a) $$\int_0^{\sqrt{3}} x\arctan x\, dx \overset{(*)}{=} \frac{x^2}{2}\arctan x\Big|_0^{\sqrt{3}} - \frac{1}{2}\int_0^{\sqrt{3}} \frac{x^2}{1+x^2}\, dx$$

$$= \frac{x^2}{2}\arctan x\Big|_0^{\sqrt{3}} - \frac{1}{2}\int_0^{\sqrt{3}} \frac{x^2+1-1}{1+x^2}\, dx = \frac{x^2}{2}\arctan x\Big|_0^{\sqrt{3}} - \frac{1}{2}\int_0^{\sqrt{3}} \frac{x^2+1}{1+x^2} - \frac{1}{x^2+1}\, dx$$

$$= \frac{x^2}{2}\arctan x\Big|_0^{\sqrt{3}} - \frac{1}{2}\int_0^{\sqrt{3}} 1 - \frac{1}{1+x^2}\, dx = \frac{x^2}{2}\arctan x\Big|_0^{\sqrt{3}} - \frac{1}{2}(x-\arctan x)\Big|_0^{\sqrt{3}}$$

$$= \frac{x^2}{2}\arctan x\Big|_0^{\sqrt{3}} - \frac{1}{2}x + \frac{1}{2}\arctan x\Big|_0^{\sqrt{3}}$$

$$= \frac{1}{2}\left(3\arctan\sqrt{3} - 0\arctan 0\right) - \frac{1}{2}\sqrt{3} + 0 + \frac{1}{2}\arctan\sqrt{3} - \frac{1}{2}\arctan 0$$

$$= \frac{3}{2}\left(\frac{\pi}{3}\right) - 0 - \frac{\sqrt{3}}{2} + 0 + \frac{1}{2}\left(\frac{\pi}{3}\right) - 0 = \boxed{\frac{2\pi}{3} - \frac{\sqrt{3}}{2}}$$

(*) I.B.P.

$u = \arctan x$	$dv = x dx$
$du = \frac{1}{1+x^2}dx$	$v = \frac{x^2}{2}$

(b) $$\int_2^{2\sqrt{3}} \frac{x^2}{\sqrt{16-x^2}}\, dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(4\sin\theta)^2}{\sqrt{16-16\sin^2\theta}} 4\cos\theta\, d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{16\sin^2\theta}{\sqrt{16(1-\sin^2\theta)}} 4\cos\theta\, d\theta = 16\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin^2\theta}{\sqrt{16}\sqrt{\cos^2\theta}} 4\cos\theta\, d\theta$$

$$= 16\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin^2\theta}{4\cos\theta} 4\cos\theta\, d\theta = 16\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin^2\theta\, d\theta$$

$$= 16\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1-\cos(2\theta)}{2}\ d\theta = 8\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 1-\cos(2\theta)\ d\theta$$

$$= 8\left(\theta - \frac{\sin(2\theta)}{2}\right)\Bigg|_{\frac{\pi}{6}}^{\frac{\pi}{3}} = 8\left(\left(\frac{\pi}{3} - \frac{\sin\left(\frac{2\pi}{3}\right)}{2}\right) - \left(\frac{\pi}{6} - \frac{\sin\left(\frac{\pi}{3}\right)}{2}\right)\right)$$

$$= 8\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4} - \frac{\pi}{6} + \frac{\sqrt{3}}{4}\right) = 8\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \frac{8\pi}{6} = \boxed{\frac{4\pi}{3}}$$

Trig. Substitute $\boxed{\begin{array}{l} x = 4\sin\theta \\ dx = 4\cos\theta d\theta \end{array}}$

$$\boxed{\begin{array}{l} x = 2 \Rightarrow x = \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6} \\ x = 2\sqrt{3} \Rightarrow x = \arcsin\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3} \end{array}}$$

OR if you don't change your limits...

$$\ldots = 8\left(\theta - \frac{\sin(2\theta)}{2}\right)\Bigg|_{x=2}^{x=2\sqrt{3}} = 8\left(\theta - \frac{2\sin\theta\cos\theta}{2}\right)\Bigg|_{x=2}^{x=2\sqrt{3}} = 8\left(\theta - \sin\theta\cos\theta\right)\Bigg|_{x=2}^{x=2\sqrt{3}}$$

$$= 8\left(\arcsin\left(\frac{x}{4}\right) - \frac{x}{4}\left(\frac{\sqrt{16-x^2}}{4}\right)\right)\Bigg|_{x=2}^{x=2\sqrt{3}} \ldots$$

(c) $\displaystyle\int_0^{\frac{\pi}{3}} \frac{\cos x}{9+4\sin^2 x}\ dx = \int_0^{\frac{\pi}{3}} \frac{\cos x}{9+(2\sin x)^2}\ dx$

$$= \frac{1}{2}\int_0^{\sqrt{3}} \frac{1}{9+u^2}\ du = \frac{1}{2}\left(\frac{1}{3}\arctan\left(\frac{u}{3}\right)\right)\Bigg|_0^{\sqrt{3}}$$

$$= \frac{1}{6}\left(\arctan\left(\frac{\sqrt{3}}{3}\right) - \arctan 0\right) = \frac{1}{6}\left(\arctan\left(\frac{1}{\sqrt{3}}\right) - \arctan 0\right) = \frac{1}{6}\left(\frac{\pi}{6} - 0\right) = \boxed{\frac{\pi}{36}}$$

Substitute $\boxed{\begin{array}{l} u = 2\sin x \\ du = 2\cos x dx \\ \frac{1}{2}du = \cos x dx \end{array}}$ $\boxed{\begin{array}{l} x = 0 \Rightarrow u = 2\sin 0 = 0 \\ x = 0 \Rightarrow u = 2\sin\frac{\pi}{3} = \sqrt{3} \end{array}}$

(d) $\displaystyle\int_1^e \left[\ln(x^3)\right]^2\ dx \overset{(*)}{=} x(\ln(x^3))^2\Big|_1^e - 6\int_1^e \ln(x^3)\ dx$

$$\overset{(**)}{=} x(\ln(x^3))^2\Big|_1^e - 6\left(x\ln(x^3)\Big|_1^e - \int_1^e 3\ dx\right)$$

$$= x(\ln(x^3))^2\Big|_1^e - 6x\ln(x^3)\Big|_1^e + 18x\Big|_1^e = x(\ln(x^3))^2 - 6x\ln(x^3) + 18x\Big|_1^e$$

$$= e\left(\ln\left(e^3\right)\right)^2 - 6e\ln\left(e^3\right) + 18e - (\ln 1 - 6\ln 1 + 18)$$

$$= 9e - 18e + 18e - 18 = 9e - 18 = \boxed{9e - 18}$$

(*) I.B.P. $\boxed{\begin{array}{ll} u = (\ln(x^3))^2 & dv = dx \\ du = 2\ln(x^3)\left(\frac{1}{x^3}\right)(3x^2)dx = \frac{6\ln(x^3)}{x}dx & v = x \end{array}}$

(**) I.B.P. $\boxed{\begin{array}{ll} u = \ln(x^3) & dv = dx \\ du = \frac{3x^2}{x^3}dx = \frac{3}{x}dx & v = x \end{array}}$

4. [15 Points] Compute the following **indefinite integral**.

$$\int \frac{\cos x}{\left(1+\sin^2 x\right)^{\frac{7}{2}}}\ dx = \int \frac{1}{[1+u^2]^{\frac{7}{2}}}\ du = \int \frac{1}{(1+\tan^2\theta)^{\frac{7}{2}}} \cdot \sec^2\theta\ d\theta$$

$$= \int \frac{1}{(\sec^2\theta)^{\frac{7}{2}}} \cdot \sec^2\theta\ d\theta = \int \frac{1}{(\sqrt{\sec^2\theta})^7} \cdot \sec^2\theta\ d\theta = \int \frac{1}{(\sec\theta)^7} \cdot \sec^2\theta\ d\theta$$

$$= \int \frac{\sec^2\theta}{\sec^7\theta}\ d\theta = \int \frac{1}{\sec^5\theta}\ d\theta = \int \cos^5\theta\ d\theta$$

$$= \int \cos^4\theta\cos\theta\ d\theta = \int (1-\sin^2\theta)^2\cos\theta\ d\theta$$

$$= \int (1-w^2)^2\ dw = \int 1 - 2w^2 + w^4\ dw$$

$$= w - \frac{2w^3}{3} + \frac{w^5}{5} + C = \sin\theta - \frac{2\sin^3\theta}{3} + \frac{\sin^5\theta}{5} + C$$

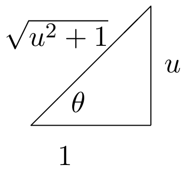
$$= \frac{u}{\sqrt{1+u^2}} - \frac{2\left(\frac{u}{\sqrt{1+u^2}}\right)^3}{3} + \frac{\left(\frac{u}{\sqrt{1+u^2}}\right)^5}{5} + C$$

$$= \frac{\sin x}{\sqrt{1+(\sin x)^2}} - \frac{2\left(\frac{\sin x}{\sqrt{1+(\sin x)^2}}\right)^3}{3} + \frac{\left(\frac{\sin x}{\sqrt{1+(\sin x)^2}}\right)^5}{5} + C$$

$$= \boxed{\frac{\sin x}{\sqrt{1+\sin^2 x)}} - \frac{2\sin^3 x}{3\left(1+\sin^2 x\right)^{\frac{3}{2}}} + \frac{\sin^5 x}{5\left(1+\sin^2 x\right)^{\frac{5}{2}}} + C}$$

Standard u substitution to simplify at the start: $u = \sin x$, $du = \cos x \; dx$

Trig. Substitute $u = \tan\theta$, $du = \sec^2\theta d\theta$



Standard w substitution for *odd* trig. integral $\int \cos^5\theta \; d\theta$ technique:

$w = \sin\theta$, $dw = \cos\theta \; d\theta$

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OPTIONAL BONUS

Do not attempt this unless you are completely done with the rest of the exam.

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OPTIONAL BONUS #1

Compute the following **indefinite integral**.

1. $$\int \frac{x^4 - 8x^3 + 24x^2 - 32x + 16}{(4x - x^2)^{\frac{7}{2}}} \; dx$$

Complete the square and factor $$= \int \frac{(x-2)^4}{(4-(x-2)^2)^{\frac{7}{2}}} \; dx$$

Standard u substitution to simplify at the start: $u = x - 2$, $du = \; dx$

$$= \int \frac{u^4}{(4-u^2)^{\frac{7}{2}}} \; dx = \int \frac{16\sin^4\theta}{(4 - 4\sin^2\theta)^{\frac{7}{2}}} 2\cos\theta \; d\theta$$

$$= \int \frac{16\sin^4\theta}{(4\cos^2\theta)^{\frac{7}{2}}} 2\cos\theta \; d\theta = \int \frac{16\sin^4\theta}{(2\cos\theta)^7} 2\cos\theta \; d\theta$$

$$= \frac{1}{4}\int \frac{\sin^4\theta}{\cos^7\theta}\cos\theta \; d\theta = \frac{1}{4}\int \frac{\sin^4\theta}{\cos^6\theta} \; d\theta$$

$$= \frac{1}{4}\int \frac{\sin^4\theta}{\cos^4\theta}\frac{1}{\cos^2\theta} \; d\theta = \frac{1}{4}\int \tan^4\theta\sec^2\theta \; d\theta$$

$$= \frac{1}{4}\left(\frac{\tan^5\theta}{5}\right) + C = \frac{1}{4}\left(\frac{\left(\frac{u}{\sqrt{4-u^2}}\right)^5}{5}\right) + C$$

$$= \frac{1}{4}\left(\frac{\left(\frac{x-2}{\sqrt{4-(x-2)^2}}\right)^5}{5}\right) + C = \boxed{\frac{1}{20}\left(\frac{(x-2)^5}{(4-(x-2)^2)^{\frac{5}{2}}}\right) + C}$$

Trig. Substitute $\boxed{\begin{array}{l} u = 2\sin\theta \\ \\ du = 2\cos\theta d\theta \end{array}}$