Name: Solutions

Amherst College

DEPARTMENT OF MATHEMATICS

Math 121

Midterm Exam #1

September 30, 2022

• This is a closed-book examination. No books, notes, calculators, cell phones, communication devices of any sort, or other aids are permitted.

• Numerical answers such as $\sin\left(\frac{\pi}{6}\right)$, $4^{\frac{3}{2}}$, $\arctan(\sqrt{3})$, $e^{\ln 4}$, $\ln(e^7)$, or $e^{3\ln 3}$ should be simplified.

• Please *show* all of your work and *justify* all of your answers. (You may use the backs of pages for additional work space.)

Problem	Score	Possible Points
1		30
2		14
3		10
4		10
5		12
6		12
7		12
Total		100

1. Limits [30 Points total, 10 Points each] Evaluate each of the following. Please justify/simplify.

(a) Show that $\lim_{x\to 0} \dfrac{\cos(3x) - \arctan(2x) + 2x - 1}{e^{-4x} - 1 + 4x} = \boxed{-\dfrac{9}{16}}$ ✓ $\dfrac{0}{0}$

$$\overset{L'H}{=} \lim_{x\to 0} \frac{-\sin(3x)\cdot 3 - \frac{1}{1+(2x)^2}\cdot 2 + 2}{-4e^{-4x}+4} \qquad \frac{-1\cdot 2+2}{-4+4} = \frac{0}{0}$$

$$\overset{L'H}{=} \lim_{x\to 0} \frac{-\cos(3x)\cdot 9 - \frac{-1}{(1+4x^2)^2}\cdot \overset{8x}{\frac{d}{dx}(1+4x^2)}\cdot 2 + 0}{16e^{-4x} + 0}$$

$$= \lim_{x\to 0} \frac{-9\cos(3x) + \frac{16x}{(1+4x^2)^2}}{16e^{-4x}} = \boxed{-\frac{9}{16}}$$

1. Limits (Continued) Evaluate each of the following. Please justify/simplify.

(b) Show that $\lim_{x\to 0^+} x\ln x = \boxed{0}$ ✓ $0^+ \cdot \ln(0^+) = 0\cdot(-\infty)$ indet.

$$= \lim_{x\to 0^+} \frac{\ln(x)}{1/x} \qquad \frac{\infty}{\infty}$$

$$\overset{L'H}{=} \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} \frac{-1}{\cancel{x}}\cdot\frac{x^{\cancel{2}}}{1} = \lim_{x\to 0^+}(-x)$$

$$= \boxed{0}.$$

(c) Show that $\lim_{x\to\infty}\left(1-\arcsin\left(\frac{2}{x^6}\right)\right)^{x^6} = \boxed{e^{-2}}$ ✓ 1^∞.

$$= e^{\lim_{n\to\infty} x^6 \ln\left(1-\arcsin\left(\frac{2}{x^6}\right)\right)}$$

Limit in the exponent:

$$\lim_{x\to\infty} x^6 \ln\left(1-\arcsin\left(\frac{2}{x^6}\right)\right) \qquad \infty\cdot 0$$

$$= \lim_{x\to\infty} \frac{\ln\left(1-\arcsin\left(\frac{2}{x^6}\right)\right)}{1/x^6} \qquad \frac{0}{0}$$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{1-\arcsin(2/x^6)}\cdot\frac{d}{dx}\left(1-\arcsin\left(\frac{2}{x^6}\right)\right)}{-6/x^7}.$$

$\frac{d}{dx}\left(\frac{2}{x^6}\right) = 2\cdot(-6)\cdot x^{-7} = -12/x^7$

$$= \lim_{x\to\infty} \frac{\frac{1}{1-\arcsin(2/x^6)}\cdot(-1)\cdot\frac{1}{\sqrt{1-(2/x^6)^2}}\cdot(-12)\cdot\frac{1}{\cancel{x^7}}}{-6/\cancel{x^7}}\cdot\frac{\cancel{x^7}}{\cancel{x^7}}$$

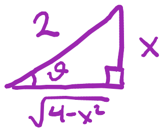
$$= \frac{1\cdot(-1)\cdot 1\cdot(-12)}{-6} = \frac{12}{-6} = -2$$

orig. limit is therefore $\boxed{e^{-2}}$.

Integrals [34 Points total] Compute the following Definite Integral. Please justify/simplify.

2. Show that $\int_{-2}^{2} \sqrt{4-x^2}\, dx = \boxed{2\pi}$

Trig. sub.



$$\left\{\begin{array}{l} x = 2\sin\theta \\ dx = 2\cos\theta\, d\theta \\ \sqrt{4-x^2} = 2\cos\theta \\ \theta = \arcsin(x/2). \end{array}\right.$$

$$= \int_{\arcsin(-1)}^{\arcsin(1)} 2\cos\theta \cdot 2\cos\theta\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} 4\cos^2\theta\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} 2(1+\cos 2\theta)\, d\theta$$

$$= \left[2\theta + \sin 2\theta\right]_{-\pi/2}^{\pi/2}$$

$$= \left(2\cdot\frac{\pi}{2} + \cancel{\sin\pi}^{\,0}\right) - \left(2\left(-\frac{\pi}{2}\right) + \cancel{\sin(-\pi)}^{\,0}\right)$$

$$= \pi + \pi$$

$$= \boxed{2\pi}.$$

Integrals (Continued) Compute each of the following Definite Integrals. Please justify/simplify.

3. Show that $\displaystyle\int_0^{\ln\sqrt{3}} \frac{e^x}{\sqrt{4-e^{2x}}}\,dx = \boxed{\frac{\pi}{6}}$ ✓

$e^{2x} = (e^x)^2$

$$\begin{cases} u = e^x \\ du = e^x dx \end{cases}$$

$$= \int_{e^0}^{e^{\ln\sqrt{3}}} \frac{du}{\sqrt{4-u^2}} = \int_1^{\sqrt{3}} \frac{1}{\sqrt{4-u^2}} \cdot \frac{1/2}{1/2}\, du$$

$$= \int_1^{\sqrt{3}} \frac{1}{2\sqrt{1-(u/2)^2}}\, du \qquad \begin{cases} v = u/2 \\ dv = du/2 \end{cases}$$

dv

$$= \int_{1/2}^{\sqrt{3}/2} \frac{dv}{\sqrt{1-v^2}} = \arcsin v \Big|_{1/2}^{\sqrt{3}/2}$$

$$= \arcsin\frac{\sqrt{3}}{2} - \arcsin\frac{1}{2} = \frac{\pi}{3} - \frac{\pi}{6} = \boxed{\frac{\pi}{6}}.$$

4. Show that $\displaystyle\int_e^{e^3} \frac{1}{x[3+(\ln x)^2]}\,dx = \boxed{\frac{\pi}{6\sqrt{3}}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\pi\sqrt{3}}{6\cdot 3} = \frac{\pi\sqrt{3}}{18}$ ✓

$$\begin{cases} u = \ln x \\ du = \frac{1}{x}\, dx \end{cases}$$

$$= \int_1^3 \frac{du}{3+u^2} \cdot \frac{1/3}{1/3} = \int_1^3 \frac{1/3}{1+(u/\sqrt{3})^2}\, du$$

$$\begin{cases} v = u/\sqrt{3} \\ dv = du/\sqrt{3} \end{cases} \rightarrow du = \sqrt{3}\cdot dv$$

$$= \int_{1/\sqrt{3}}^{\sqrt{3}} \frac{1/3}{1+v^2} \sqrt{3}\, dv$$

$$= \frac{\sqrt{3}}{3} \cdot \arctan v \Big|_{1/\sqrt{3}}^{\sqrt{3}}$$

$$= \frac{\sqrt{3}}{3}\left(\arctan\sqrt{3} - \arctan\frac{1}{\sqrt{3}}\right) = \frac{\sqrt{3}}{3}\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \frac{\sqrt{3}}{3}\cdot\frac{\pi}{6} = \boxed{\frac{\pi\sqrt{3}}{18}}$$

$\sqrt{3}$ 1 $\sqrt{3}$ 1

More Integrals [36 Points total] Compute the following Indefinite Integral. Please justify/simplify.

5. Compute $\int \underbrace{x}_{A} \underbrace{\arcsin x}_{I} \, dx$

$$\begin{cases} u = \arcsin x & dv = x\,dx \\ du = \frac{1}{\sqrt{1-x^2}}\,dx & v = \frac{1}{2}x^2 \end{cases}$$

$$= \frac{1}{2}x^2 \arcsin x - \int \frac{1}{2} \cdot \frac{x^2}{\sqrt{1-x^2}}\,dx$$

Trig. sub.

$$\begin{cases} x = \sin\theta \\ dx = \cos\theta\,d\theta \\ \sqrt{1-x^2} = \cos\theta \\ \theta = \arcsin x \end{cases}$$

$$= \frac{1}{2}x^2 \arcsin x - \frac{1}{2}\int \frac{\sin^2\theta}{\cancel{\cos\theta}} \cdot \cancel{\cos\theta}\,d\theta$$

$$= \frac{1}{2}x^2 \arcsin x - \frac{1}{4}\int (1 - \cos 2\theta)\,d\theta$$

$$= \frac{1}{2}x^2 \arcsin x - \frac{1}{4}\theta + \frac{1}{4} \cdot \frac{1}{\cancel{2}} \underbrace{\sin 2\theta}_{\cancel{2}\sin\theta\cos\theta} + C$$

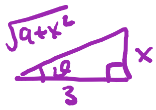
$$= \underline{\frac{1}{2}x^2 \arcsin x - \frac{1}{4}\arcsin x + \frac{1}{4}x\sqrt{1-x^2} + C.}$$

More Integrals (Continued) Compute the following Indefinite Integral. Please justify/simplify.

6. Compute $\displaystyle\int \frac{1}{(9+x^2)^{\frac{7}{2}}}\,dx$ Hint: $3^6 = 729$

$(\sqrt{9+x^2})^7$

Trig sub.



$x = 3\tan\theta$

$dx = 3\sec^2\theta\, d\theta$

$\sqrt{9+x^2} = 3\sec\theta$

$\theta = \arctan(x/3)$.

$$= \int \frac{1}{(3\sec\theta)^7}\cdot 3\sec^2\theta\, d\theta$$

odd!

$$= \int \frac{3}{3^7}\cdot\frac{\sec^2\theta}{\sec^7\theta}\,d\theta = \frac{1}{3^6}\int \cos^5\theta\, d\theta$$

$\frac{1}{\sec^5\theta}$

$$= \frac{1}{3^6}\int (1-\sin^2\theta)^2\cos\theta\, d\theta$$

$u = \sin\theta$

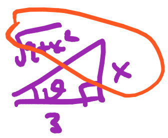
$du = \cos\theta\, d\theta$

$$= \frac{1}{3^6}\int (1-u^2)^2\,du = \frac{1}{3^6}\int (1-2u^2+u^4)\,du$$

$$= \frac{1}{3^6}\left[u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right] + C$$

$$= \frac{1}{3^6}\left[\sin\theta - \frac{2}{3}\sin^3\theta + \frac{1}{5}\sin^5\theta\right] + C$$

$\sin\theta = ?$ in terms of x?



$= \dfrac{x}{\sqrt{9+x^2}}$

plug this in:

$$\frac{1}{729}\left(\frac{x}{\sqrt{9+x^2}} - \frac{2}{3}\frac{x^3}{(9+x^2)^{3/2}} + \frac{1}{5}\frac{x^5}{(9+x^2)^{5/2}}\right) + C$$

More Integrals (Continued) Compute the following Indefinite Integral. Please justify/simplify.

7. Compute $\int \underbrace{x^7}_{A} \cdot \underbrace{\ln\left(x^3\right)}_{L}\, dx$ $= \int 3x^7 \ln x \, dx$

IBP: $\begin{cases} u = \ln x & dv = 3x^7\, dx \\ du = \frac{1}{x} dx & v = \frac{3}{8}x^8 \end{cases}$

$= \frac{3}{8}x^8 \ln x - \int \frac{3}{8} x^{\cancel{8}\,7} \cdot \frac{1}{\cancel{x}}\, dx$

$= \frac{3}{8}x^8 \ln x - \frac{3}{8} \cdot \frac{1}{8} x^8 + C$

$= \underline{\frac{3}{8}x^8 \ln x - \frac{3}{64}x^8 + C}$.