Goal Verify convergnece using the Integral Test, p-series, Comparison and Limit Comparison Test. **Reference:** §11.3, 11.4

Examples to study first

In each example: **Determine whether the given Series Converges or Diverges. Justify with any Convergence Test(s).**

Example $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n}$

Solution The related Function $f(x) = \dfrac{\ln x}{x}$ is continuous $(x > 0)$, positive $(x > 1)$, and decreasing for $x > e$ since $f'(x) = \dfrac{1 - \ln x}{x^2} < 0$.
Therefore, we study the Related Integral

$$\int_2^{\infty} \frac{\ln x}{x}\, dx = \lim_{t\to\infty} \int_2^t \frac{\ln x}{x}\, dx = \lim_{t\to\infty} \int_{\ln 2}^{\ln t} u\, du = \lim_{t\to\infty} \frac{u^2}{2}\bigg|_{\ln 2}^{\ln t} = \lim_{t\to\infty} \frac{\cancelto{\infty}{(\ln t)^2}}{2} - \frac{(\ln 2)^2}{2} = \boxed{\infty}$$

$u = \ln x$
$du = \frac{1}{x}\, dx$

$x = 2 \Rightarrow u = \ln 2$
$x = t \Rightarrow u = \ln t$

Therefore, the Improper Integal Diverges. As a result, the Original Series also **Diverges by the Integral Test**.

Example $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3 + 7}$

Solution Observe that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3+7} \approx \sum_{n=1}^{\infty} \frac{1}{n^3}$, so we will use $\sum \frac{1}{n^3}$ as a comparison series. It is convergent since it is a p-series with $p = 3 > 1$. We can bound the terms with $\dfrac{1}{n^3+1} \leq \dfrac{1}{n^3}$. Therefore the original series also converges, by the comparison test.

Example

Solution $\displaystyle\sum_{n=1}^{\infty} \frac{n^3+2}{n^4+8} \approx \sum_{n=1}^{\infty} \frac{n^3}{n^4} = \sum_{n=1}^{\infty} \frac{1}{n}$ $\leftarrow$ Comparison Series: Divergent p-series $p = 1$

Next check: $\displaystyle\lim_{n\to\infty} \frac{\frac{n^3+2}{n^4+8}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^4+2n}{n^4+8} \frac{\left(\frac{1}{n^4}\right)}{\left(\frac{1}{n^4}\right)} = \lim_{n\to\infty} \frac{1 + \cancelto{0}{\frac{2}{n^3}}}{1 + \cancelto{0}{\frac{8}{n^4}}} = 1$ Finite and Non-Zero

Therefore, the Original Series also **Diverges by the Limit Comparision Test**.

Problems to hand in

> **Note** Much of this problem set concerns the comparison test (CT) and limit comparison test (LCT), which we'll cover on Monday 3/20. You will probably want to wait until then to work on these (especially since you should be resting during break!)

Use the Integral Test to determine whether the given series Converges or Diverges. You do **NOT** need to check the preconditions (continuous, postive, decreasing) for the Integral Test this time.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ 2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ 3. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ 4. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{e^n}$

5. Consider $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+4}$. Use **two** Different methods, namely the Integral Test (no pre-Condition check needed) and the Comparison Test, to prove that this series Converges.

Determine if the series Converges or Diverges using either the Comparison **OR** Limit Comparison Test.

6. $\displaystyle\sum_{n=1}^{\infty} \frac{9^n}{3+10^n}$ 7. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+5}{n^3}$ 8. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{\sqrt{n}+2}$ 9. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+7}{n^7+2}$

10. Consider $\displaystyle\sum_{n=1}^{\infty} \frac{5n^2+n}{n^4}$. Use **two** Different methods to prove that this series Converges. Use the Limit Comparison Test and then a *split-split* algebra technique into p-series pieces.

Determine whether the given series Converges or Diverges. Justify.

11. $\displaystyle\sum_{n=1}^{\infty} \sin^2\left(\frac{\pi n^4+1}{6n^4+5}\right)$ 12. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2\left(\pi n^4+1\right)}{6n^4+5}$ 13. $\displaystyle\sum_{n=1}^{\infty} \frac{7}{n^9}+\frac{7^n}{9^n}$

REVIEW

14. $\displaystyle\sum_{n=1}^{\infty} n^6+6$ 15. $\displaystyle\sum_{n=1}^{\infty} \frac{n^6+6}{n^6+1}$ 16. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^6+1}$