Goal Apply absolute convergence test (ACT), alternating series test (AST), and ratio test (RT) to analyze convergence. Classify series with positive and negative terms as absolutely convergent, conditionally convergent, or divergent.
Reference: §11.5, 11.6, 11.7

Examples to study first

In each example, **Determine whether the Series is Absolutely Convergent, Conditionally Convergent, or Divergent.**

Example $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^2+7}{n^7+2}$

Solution The absolute series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+7}{n^7+2} \approx \sum_{n=1}^{\infty} \frac{n^2}{n^7} = \sum_{n=1}^{\infty} \frac{1}{n^5}$. This comparison series converges by the p-series test ($p = 5 > 1$).

Check: $\displaystyle\lim_{n\to\infty} \frac{\frac{n^2+7}{n^7+2}}{\frac{1}{n^5}} = \lim_{n\to\infty} \frac{n^7+7n^5}{n^7+2} \cdot \frac{\frac{1}{n^7}}{\frac{1}{n^7}} = \lim_{n\to\infty} \frac{1+\frac{7}{n^2}}{1+\frac{2}{n^7}} = 1$. This is finite and non-zero, so we can use LCT. Since the comparison series converges, the **Absolute Series** $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+7}{n^7+2}$ also **Converges** by Limit Comparison Test (LCT). Finally, the Original Series is $\boxed{\text{Absolutely Convergent (A.C.)}}$ (by Definition).

Example $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{7n+3}$

Solution The absolute series is $\sum_{n=1}^{\infty} \frac{1}{7n+3} \approx \sum_{n=1}^{\infty} \frac{1}{n}$. This comparison seires is divergent by p-Series ($p = 1$). We'll use the LCT.

Check: $\lim_{n\to\infty} \frac{\frac{1}{7n+3}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{7n+3} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{7+\frac{3}{n}} = \frac{1}{7} \neq 0.$

Therefore, the **absolute Series** also **diverges** by Limit Comparison Test.
Now, we must examine the original alternating series with the Alternating Series Test. We must check the three criteria:

1. $b_n = \frac{1}{7n+3} > 0$
2. $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\cancel{7n+3} \to \infty} = 0$
3. Terms Decreasing: $\frac{1}{b_{n+1}} < \frac{1}{b_n}$ because $b_{n+1} = \frac{1}{7(n+1)+3} = \frac{1}{7n+10} < \frac{1}{7n+3} = b_n$

Therefore, the **Original Series Converges** by the Alternating Series Test. Finally, we can conclude the Original Series is $\boxed{\text{Conditionally Convergent (C.C.)}}$ (by Definition).

Example

Solution $\sum_{n=1}^{\infty} \frac{n^n}{n! \cdot 2^n}$ Try Ratio Test:

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(n+1)^{n+1}}{(n+1)!2^{n+1}}}{\frac{n^n}{n! \cdot 2^n}} \right| = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{n^n} \cdot \frac{n!}{(n+1)!} \cdot \frac{2^n}{2^{n+1}}$$

$$= \lim_{n\to\infty} \frac{(n+1)^n \cancel{(n+1)}}{n^n} \cdot \frac{\cancel{n!}}{\cancel{(n+1)}\cancel{n!}} \cdot \frac{\cancel{2^n}}{\cancel{2^n} \cdot 2} = \lim_{n\to\infty} \cancel{\left(\frac{n+1}{n}\right)^n} \to e \cdot \left(\frac{1}{2}\right) = \frac{e}{2} > 1$$

The Original Series $\boxed{\text{Diverges by the Ratio Test}}$.

Example

Solution $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (2n)!}{e^{2n} \cdot\ n! \cdot\ n^n}$ Try Ratio Test:

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\dfrac{(-1)^{n+1}(2(n+1))!}{e^{2(n+1)}\ (n+1)!\ (n+1)^{n+1}}}{\dfrac{(-1)^n(2n)!}{e^{2n}\ n!\ n^n}} \right|$$

$$= \lim_{n\to\infty} \frac{(2n+2)!}{(2n)!} \cdot \frac{n^n}{(n+1)^{n+1}} \cdot \frac{e^{2n}}{e^{2n+2}} \cdot \frac{n!}{(n+1)!}$$

$$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)(2n)!}{(2n)!} \cdot \frac{n^n}{(n+1)^n(n+1)} \cdot \frac{\cancel{e^{2n}}}{\cancel{e^{2n}}e^2} \cdot \frac{\cancel{n!}}{(n+1)\cancel{n!}}$$

$$= \lim_{n\to\infty} \frac{(2\cancel{(n+1)})(2n+1)}{\cancel{(n+1)}(n+1)} \cdot \left(\frac{n}{n+1}\right)^{n} \cdot \frac{1}{e^2} \qquad \left(\frac{n}{n+1}\right)^n \to \frac{1}{e}$$

$$= \lim_{n\to\infty} \frac{2n+1}{n+1} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} \cdot \left(\frac{2}{e^3}\right) = \lim_{n\to\infty} \frac{2+\frac{1}{n}}{1+\frac{1}{n}} \cdot \left(\frac{2}{e^3}\right) = 2\left(\frac{2}{e^3}\right) = \frac{4}{e^3} < 1 \qquad \left(\frac{1}{n}\to 0\right)$$

The original series is Absolutely Convergent (A.C.) by the Ratio Test.

Problems to hand in

Determine whether each series is Absolutely Convergent, Conditionally Convergent or Divergent.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3+1}$
2. $\displaystyle\sum_{n=1}^{\infty} (-1)^n\ \frac{n^3+8}{n^8+3}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{5n+1}$
4. $\displaystyle\sum_{n=1}^{\infty} (-1)^n\ \frac{n+1}{n^2}$
5. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^3}{n^7+2}$
6. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \sin^2 n}{n^8+2}$

Determine whether each series is convergent or divergent. Suggestion: use the **ratio test** for these.

7. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{5^n}$
8. $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n}{(2n+1)!}$
9. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{100^n}$
10. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$
11. $\displaystyle\sum_{n=1}^{\infty} \frac{n^{100}\ 100^n}{n!}$
12. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$
13. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\ (\ln n)\ (2n)!}{n^n\ 2^{3n}\ (n!)}$

14. Consider the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$.

(a) Show that n^{th} Term Divergence Test is **Inconclusive**.

(b) Show that the Ratio Test is **Inconclusive**.

(c) Show that the series Diverges using the Integral Test. Skip checking the 3 preconditions here. **Note:** This is an example where the terms approach 0 but the series Diverges.