Goal power series, especially interval and radius of convergence.
Reference: §11.8

Examples to study first

In each example, **Determine the Interval and Radius of Convergence. Justify.**

Example $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(5x-2)^n}{(n+5)\ 8^n}$.

Solution Use the Ratio Test. $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

$$= \lim_{n\to\infty} \left| \frac{\dfrac{\cancel{(-1)^{n+1}}(5x-2)^{n+1}}{(n+6)8^{n+1}}}{\dfrac{\cancel{(-1)^n}(5x-2)^n}{(n+5)8^n}} \right| = \lim_{n\to\infty} \left|\frac{(5x-2)^{n+1}}{(5x-2)^n}\right| \cdot \left(\cancelto{1}{\frac{n+5}{n+6}}\right) \cdot \frac{8^n}{8^{n+1}} = \frac{|5x-2|}{8}$$

The Ratio Test gives convergence for x when $\frac{|5x-2|}{8} < 1$ or $|5x-2| < 8$.

That is $-8 < 5x-2 < 8 \Longrightarrow -6 < 5x < 10 \Longrightarrow -\frac{6}{5} < x < 2$

Manually Test Endpoints: (where $L = 1$ and Ratio Test is Inconclusive)

$\bullet x = 2$ The original series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(5(2)-2)^n}{(n+5)\ 8^n} = \sum_{n=1}^{\infty} \frac{(-1)^n\cancel{8^n}}{(n+5)\ \cancel{8^n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n+5}$

which is Convergent by AST: 1. $b_n = \frac{1}{n+5} > 0$ 2. $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n+5} = 0$

3. Terms decreasing: $b_{n+1} = \frac{1}{n+6} < \frac{1}{n+5} = b_n \Rightarrow$ $x = 2$ is Included in the Domain.

$\bullet x = -\frac{6}{5}$ The original series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\left(5\left(-\frac{6}{5}\right)-2\right)^n}{(n+5)\ 8^n} = \sum_{n=1}^{\infty} \frac{(-1)^n(-8)^n}{(n+5)\ 8^n}$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n\cancel{8^n}}{(n+5)\cancel{8^n}} = \sum_{n=1}^{\infty} \frac{\cancelto{1}{(-1)^{2n}}}{n+5} = \sum_{n=1}^{\infty} \frac{1}{n+5} \approx \sum_{n=1}^{\infty} \frac{1}{n} \text{ the Div Harmonic } p\text{-Series } p = 1.$$

LCT: $\displaystyle\lim_{n\to\infty} \frac{\frac{1}{n+5}}{\frac{1}{n}} = \lim_{n\to\infty} \cancelto{1}{\frac{n}{n+5}} = 1$ which is *Finite* and *Non-zero*. Therefore, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+5}$ is also Divergent by LCT $\Rightarrow$ $x = -\frac{6}{5}$ is NOT included in the Domain.

Finally, Interval of Convergence $\boxed{I = \left(-\frac{6}{5}, 2\right]}$ with Radius of Convergence $\boxed{R = \frac{8}{5}}$.

Example $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$.

Solution Use Ratio Test. $\displaystyle L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

$$= \lim_{n\to\infty} \left| \frac{\dfrac{x^{2(n+1)+1}}{(2(n+1)+1)!}}{\dfrac{x^{2n+1}}{(2n+1)!}} \right| = \lim_{n\to\infty} \left|\frac{x^{2n+3}}{x^{2n+1}}\right| \frac{(2n+1)!}{(2n+3)!} = \lim_{n\to\infty} \frac{x^2}{\underbrace{(2n+3)(2n+2)}_{\to\infty}} = 0 < 1 \text{ for all } x$$

Converges by the Ratio Test for all x and $\boxed{I = (-\infty, \infty)}$ with $\boxed{R = \infty}$.

Example $\displaystyle\sum_{n=0}^{\infty} n^n (x-7)^n$.

Solution Use Ratio Test. $\displaystyle L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

$$= \lim_{n\to\infty} \left|\frac{(n+1)^{n+1}(x-7)^{n+1}}{n^n (x-7)^n}\right| = \lim_{n\to\infty} \overset{\to e}{\frac{(n+1)^n}{n^n}} \overset{\to\infty}{(n+1)} |x-7| = \infty > 1$$

Diverges by the Ratio Test for all x *unless* $x - 7 = 0$ or $x = 7$. So $\boxed{I = \{7\}}$ with $\boxed{R = 0}$.

Problems to hand in

Determine the Interval and Radius of Convergence for each of the following Power Series. Use the Ratio Test and manually check convergence at the Endpoints for the Finite Intervals. Follow the examples above for statements/format for all three cases.

1. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^4 \cdot 4^n}$
3. $\displaystyle\sum_{n=1}^{\infty} n!\ \ln n\ (x-6)^n$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\ (9x-4)^n}{n^8 \cdot 5^n}$
5. $\displaystyle\sum_{n=0}^{\infty} (3n)!\ (2x-1)^n$
6. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\ (6x+1)^n}{(6n+1) \cdot 7^n}$
7. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n\ x^{2n}}{(2n)!}$
8. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\ (3x-5)^n}{(n+6)^2 \cdot 7^{n+1}}$

Find the Power Series Representation for the following functions and determine the Interval of Convergence.

9. $\displaystyle f(x) = \frac{1}{1+x}$
10. $\displaystyle f(x) = \frac{5}{1-4x}$
11. $\displaystyle f(x) = \frac{1}{3-x}$