Goal Review exponentials and logarithms, using limits, derivatives, and integrals (that is, through the lens of calculus).

Reference Stewart §6.2-4.

Examples to study first

Many future assignments will begin with one or more "examaples to study first," like these. You do not need to hand these in, but you should study them carefully! They will help guide you through some of the problems on that weeks problem set, and provide an example of how you may organize your work.

Example Think about the graph of $y = \ln x$. We know that $\boxed{\lim_{x\to 0^+} \ln x = \lim_{x\to 0^+} \ln \overset{0^+}{x} = -\infty}$. Learn this!

Example Evaluate $\lim_{x\to 5^+} \ln(x-5)$.

Solution $\lim_{x\to 5^+} \ln(x-5) = \lim_{x\to 5^+} \ln \overset{0^+}{(\overset{5^+}{x}-5)} = -\infty.$

Example Evaluate $\lim_{x\to 8^-} \ln|x-8|$.

Solution $\lim_{x\to 8^-} \ln|x-8| = \lim_{x\to 8^-} \ln \overset{0^+}{|\overset{0^-}{\overset{8^-}{x}-8}|} = -\infty.$

Example Evaluate the **indefinite integral** $\int \frac{(3-\sqrt{x})(1+2\sqrt{x})}{x^2}\,dx$.

Solution The key insight is that you can expand this into a bunch of powers of x.

$$\begin{aligned}
\int \frac{(3-\sqrt{x})(1+2\sqrt{x})}{x^2}\,dx &= \int \frac{3+6\sqrt{x}-\sqrt{x}-2x}{x^2}\,dx = \int \frac{3+5\sqrt{x}-2x}{x^2}\,dx \\
&= \int \frac{3}{x^2} + \frac{5\sqrt{x}}{x^2} - \frac{2x}{x^2}\,dx = \int \frac{3}{x^2} + \frac{5}{x^{\frac{3}{2}}} - \frac{2}{x}\,dx \\
&= \int \frac{3}{x^2} + \frac{5}{x^{\frac{3}{2}}} - \frac{2}{x}\,dx \\
&\overset{\text{prep}}{=} \int 3x^{-2} + 5x^{-\frac{3}{2}} - \frac{2}{x}\,dx \\
&= -3x^{-1} + 5(-2)x^{-\frac{1}{2}} - 2\ln|x| + C \\
&= \boxed{-\frac{3}{x} - \frac{10}{\sqrt{x}} - 2\ln|x| + C}
\end{aligned}$$

Example Evaluate the **definite integral** $\int_{\ln 3}^{\ln 8} \frac{e^x}{\sqrt{1+e^x}}\,dx$.

Solution $\int_{\ln 3}^{\ln 8} \frac{e^x}{\sqrt{1+e^x}}\,dx = \int_4^9 \frac{1}{\sqrt{u}}\,du = \int_4^9 u^{-\frac{1}{2}}\,du = 2\sqrt{u}\Big|_4^9 = 2\sqrt{9}-2\sqrt{4} = 6-4 = \boxed{2}$

Here $\begin{array}{|rl|} \hline u &= 1+e^x \\ du &= e^x\,dx \\ \hline \end{array}$ and $\begin{array}{|ll|} \hline x = \ln 3 & \Longrightarrow u = 1+e^{\ln 3} = 1+3 = 4 \\ x = \ln 8 & \Longrightarrow u = 1+e^{\ln 8} = 1+8 = 9 \\ \hline \end{array}$

Example Evaluate the **definite integral** $\int_1^2 \frac{1}{3-5x}\,dx$

Solution $\int_1^2 \frac{1}{3-5x}\,dx = -\frac{1}{5}\int_{-2}^{-7} \frac{1}{u}\,du = -\frac{1}{5}\ln|u|\Big|_{-2}^{-7} = -\frac{1}{5}(\ln|-7| - \ln|-2|) = \boxed{-\frac{1}{5}\ln\left(\frac{7}{2}\right)}$

Here $\begin{array}{|rl|} \hline u &= 3-5x \\ du &= -5\,dx \\ -\frac{1}{5}du &= dx \\ \hline \end{array}$ and $\begin{array}{|ll|} \hline x = 1 & \Longrightarrow u = 3-5 = -2 \\ x = 2 & \Longrightarrow u = 3-10 = -7 \\ \hline \end{array}$

Problems to hand in

Differentiate the following functions, and simplify.

1. $f(x) = e^5$
2. $f(x) = e^x + x^e$
3. $y = \dfrac{1 - e^{2x}}{1 + e^{2x}}$
4. $f(x) = e^{\sin(2x)} + \sin\left(e^{2x}\right)$
5. $y = e^{\sqrt{x}}$
6. $y = x^2 e^{-\frac{1}{x}}$
7. $y = \ln\left(1 + e^{3x}\right)$
8. $f(x) = \ln\left(\dfrac{1}{x}\right) + \dfrac{1}{\ln x}$

9. **Simplify** the following expression, as a single logarithm.

$$\frac{1}{3}\ln[(x+2)^3] + \frac{1}{2}[\ln x - \ln[(x^2+3x+2)^2]]$$

Solve each of the following equations for x.

10. $e^{7-4x} = 6$
11. $\ln(3x - 10) = 2$

Evaluate each of the following **limits**.

12. $\lim\limits_{x \to 2^-} \ln|x - 2|$
13. $\lim\limits_{x \to 3^+} \ln(x^2 - 9)$

Evaluate each of the following **integrals**. Simplify and justify your answer.

14. $\displaystyle\int e^x + x^e \, dx$
15. $\displaystyle\int_0^{\ln 4} \frac{1}{e^{2x}} \, dx$
16. $\displaystyle\int \frac{(1 + e^x)^2}{e^x} \, dx$
17. $\displaystyle\int \left(e^x + e^{-x}\right)^2 \, dx$
18. $\displaystyle\int \frac{e^x}{1 + e^x} \, dx$
19. $\displaystyle\int_2^3 \frac{1}{5 - 4x} \, dx$
20. $\displaystyle\int_e^{e^3} \frac{4}{x(\ln x)^2} \, dx$

Note Future assignments will not always specifically say "simplify and justify." These words are included above for emphasis, but you should **always simplify and justify every answer** unless otherwise stated.