Goal Gain facility with calculus of $\arcsin x$ and $\arctan x$ (also known as $\sin^{-1} x$ and $\tan^{-1} x$).
Reference Stewart §6.6

Examples to study first

Example **Prove** that $\dfrac{d}{dx} \arctan x = \dfrac{1}{1+x^2}$.

Solution Let $y = \arctan x$. We want to solve for $\dfrac{dy}{dx}$.
First, we can **invert** to obtain the equation $\tan y = x$.
Differentiate this expression, using the chain rule:

$$\frac{d}{dx}(\tan y) = \frac{d}{dx}(x)$$
$$\Rightarrow \sec^2 y \frac{dy}{dx} = 1.$$

(The symbol $\Rightarrow$ means that the previous equation logically implies this equation; it is often pronounced "implies.")
Now we can **solve** for $\frac{dy}{dx}$ to obtain $\dfrac{dy}{dx} = \dfrac{1}{\sec^2 y}$.
Finally, we need to **re-express** this result in terms of x alone, not y. We can do this using the trig identity

$$\sec^2 y = 1 + \tan^2 y,$$

which implies that

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+(\tan y)^2} = \frac{1}{1+x^2}$$

as desired.

Note Another good way to finish this argument is to write that $\sec^2 y = \sec^2(\arctan x)$, and then explain why this is equal to $1 + x^2$.

Example **Prove** that $\dfrac{d}{dx}\arcsin x = \dfrac{1}{\sqrt{1-x^2}}$.

Solution Let $y = \arcsin x$. We want to solve for $\frac{dy}{dx}$. Inverting this equation gives

$$\sin y = x.$$

Differentiating this equation and solving gives

$$\begin{aligned} \frac{d}{dx}\sin y &= \frac{d}{dx}x \\ \Rightarrow \cos y\,\frac{dy}{dx} &= 1 \\ \Rightarrow \frac{dy}{dx} &= \frac{1}{\cos y}. \end{aligned}$$

Finally, use the identity $\sin^2 y + \cos^2 y = 1$ to solve for $\cos y$: $\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}$. Combining with the equation above gives

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}.$$

Note There is actually small logical gap at the end of this argument, but I have left the gap there because it is a minor point that I will not deduct points for if you omit it. The issue is that the identity $\sin^2 y + \cos^2 y = 1$ only implies that $\cos y = \pm\sqrt{1-\sin^2 y}$; the sign is ambiguous. This gap can be filled by recalling that the range of $y = \arcsin x$ is $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$, and $\cos y \geq 0$ for all such values of y. This is a subtle point, so I will not pick on it while grading.

Problems to hand in

Differentiate the following functions. Simplify.

1. $f(x) = \tan^{-1}\left(x^2\right)$
2. $f(x) = \left(\tan^{-1}(x)\right)^2$
3. $y = x\sin^{-1} x + \sqrt{1-x^2}$
4. $f(x) = \ln\left(1 - \arcsin\left(\dfrac{2}{x^4}\right)\right)$

5. Find the value of the expression $\tan\left(\sin^{-1}\left(\dfrac{2}{3}\right)\right)$

6. Simplify the expression $\sin\left(\tan^{-1} x\right)$

7. Compute the Second Derivative for $f(x) = \arctan(2x)$

8. Compute the Second Derivative for $f(x) = \arcsin(6x)$

9. **Prove** that $\dfrac{d}{dx}\sin^{-1}(3x) = \dfrac{3}{\sqrt{1-9x^2}}$

10. **Prove** that $\dfrac{d}{dx}\tan^{-1}(5x) = \dfrac{5}{1+25x^2}$

11. Use Integration to **Justify** that $\displaystyle\int \frac{1}{3+x^2}\,dx = \frac{1}{\sqrt{3}}\arctan\left(\frac{x}{\sqrt{3}}\right) + C$

Compute each of the following Integrals. Simplify.

12. $\displaystyle\int \frac{x^2}{x^2+1}\,dx$

13. $\displaystyle\int \frac{x+1}{x^2+1}\,dx$

14. $\displaystyle\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{8}{1+x^2}\,dx$

15. $\displaystyle\int_0^{\frac{1}{2}} \frac{\arcsin x}{\sqrt{1-x^2}}\,dx$

16. $\displaystyle\int \frac{1}{\sqrt{1-x^2}\cdot\sin^{-1}x}\,dx$

17. $\displaystyle\int_1^3 \frac{1}{\sqrt{x}\,(1+x)}\,dx$

18. $\displaystyle\int_0^{\ln 3} \frac{e^x}{1+e^x}\,dx$

19. $\displaystyle\int_0^{\frac{1}{2}\ln 3} \frac{e^x}{1+e^{2x}}\,dx$

20. $\displaystyle\int \frac{e^{2x}}{\sqrt{1-e^{4x}}}\,dx$

21. $\displaystyle\int_3^{3\sqrt{3}} \frac{1}{\sqrt{36-x^2}} + \frac{1}{9+x^2}\,dx$