

Goal Practice with Inverse Trigonometric Functions, and more review of limits (no L'Hôpital's Rule yet)

Reference Stewart §1.6

Examples to study first

Example The following two formulas are called “ a -rules.” You should learn them, and you can use them freely in your solutions (just write “ a -rule” so that your reader knows what you’re doing; see the next two examples for how this can be presented).

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C.$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \arcsin\left(\frac{x}{a}\right) + C.$$

These formula is valid for any choice of a . How would you prove/derive these rules?

Solution There are at least three ways, which I’ll just summarize here for now. We saw one technique in some examples in class, using the substitution $u = x/a$. You can also differentiate the right hand side of the equation, using either the chain rule or implicit differentiation.

Example Evaluate $\int_e^{e^3} \frac{1}{x(3 + (\ln x)^2)} dx$

Solution Make the substitution $\begin{cases} u = \ln x \\ du = \frac{1}{x} dx \end{cases}$ to obtain

$$\begin{aligned} \int_e^{e^3} \frac{1}{x(3 + (\ln x)^2)} dx &= \int_{\ln e}^{\ln e^3} \frac{1}{3 + u^2} du \stackrel{a\text{-rule}}{=} \frac{1}{\sqrt{3}} \arctan\left(\frac{u}{\sqrt{3}}\right) \Big|_1^3 \\ &= \frac{1}{\sqrt{3}} \left(\arctan\left(\frac{3}{\sqrt{3}}\right) - \arctan\left(\frac{1}{\sqrt{3}}\right) \right) \\ &= \frac{1}{\sqrt{3}} \left(\frac{\pi}{3} - \frac{\pi}{6} \right) = \frac{1}{\sqrt{3}} \left(\frac{2\pi}{6} - \frac{\pi}{6} \right) = \boxed{\frac{\pi}{6\sqrt{3}}}. \end{aligned}$$

Example Evaluate $\int \frac{e^{3x}}{4 + e^{3x}} dx$.

Solution $\int \frac{e^{3x}}{4 + e^{3x}} dx = \frac{1}{3} \int \frac{1}{u} du = \frac{1}{3} \ln |u| + C = \frac{1}{3} \ln |4 + e^{3x}| + C$

where we have made the substitution:

$$\begin{aligned} u &= 4 + e^{3x} \\ du &= 3e^{3x} dx \\ \frac{1}{3} du &= e^{3x} dx \end{aligned}$$

Example Evaluate $\int \frac{e^{3x}}{4 + e^{6x}}$.

Solution Substitute $\begin{matrix} u = e^{3x} \\ du = 3e^{3x} dx \\ \frac{1}{3} du = e^{3x} dx \end{matrix}$ to obtain

$$\begin{aligned} \int \frac{e^{3x}}{4 + e^{6x}} dx &= \int \frac{e^{3x}}{4 + (e^{3x})^2} dx \\ &= \frac{1}{3} \int \frac{1}{4 + u^2} du \stackrel{\text{a-rule}}{=} \frac{1}{3} \left(\frac{1}{2} \arctan \left(\frac{u}{2} \right) \right) + C = \frac{1}{6} \arctan \left(\frac{e^{3x}}{2} \right) + C \end{aligned}$$

Note The little note “a-rule” above the equals sign above is a handy and concise way to explain what fact you’re using to justify an equation. We’ll use this type of shorthand often, e.g writing $\stackrel{\text{L'H}}{=}$ for equations justified by L’Hôpital’s rule.

Problems to hand in

Compute each of the following Integrals. Simplify.

1. $\int_2^{2\sqrt{3}} \frac{1}{\sqrt{16 - x^2}} dx$
2. $\int_0^{\ln 3} \frac{e^x}{3 + e^{2x}} dx$
3. $\int_0^{\ln \sqrt{3}} \frac{e^x}{\sqrt{4 - e^{2x}}} dx$
4. $\int_4^{4\sqrt{3}} \frac{1}{16 + x^2} dx$
5. $\int \frac{x}{\sqrt{1 - x^4}} dx$
6. $\int \frac{x^2}{x^2 + 4} dx$
7. $\int \frac{2x^2 + 5}{x^2 + 1} dx$
8. $\int \frac{1}{(1 + x^2)(5 + (\arctan x)^2)} dx$
9. $\int_3^9 \frac{1}{\sqrt{x}(x + 9)} dx$
10. $\int \frac{x^2 + x + 1}{x^2 + 4} dx$

Compute each of the following Limits. Simplify. **These limits are review; they do not require l'Hôpital's rule, which will appear on PSet 5.**

11. $\lim_{x \rightarrow 5^+} \frac{1}{x-5}$

12. $\lim_{x \rightarrow 5^-} \frac{1}{x-5}$

13. $\lim_{x \rightarrow 8^+} \ln|x-8|$

14. $\lim_{x \rightarrow 8^-} \ln|x-8|$

15. $\lim_{x \rightarrow 3^+} e^{\frac{2}{x-3}}$

16. $\lim_{x \rightarrow 3^-} e^{\frac{2}{x-3}}$

17. $\lim_{x \rightarrow \infty} \ln\left(1 - \arctan\left(\frac{5}{x^4}\right)\right)$

18. $\lim_{x \rightarrow \infty} \ln\left(\frac{\pi}{2} - \arctan x\right)$

19. $\lim_{x \rightarrow 4^-} \ln|\ln|x-4||$

20. $\lim_{x \rightarrow 0^+} \arctan\left(\frac{\ln x}{5}\right)$