Goal Evaluate limits using L'Hôpital's rule.
Reference Stewart §6.8

Examples to read first

Example Evaluate $\displaystyle\lim_{x\to 0}\frac{\arcsin x+\cos(3x)-e^x}{\arctan(3x)+x^2-\sin(3x)}$.

Solution
This limit looks like a bit of a bear, and it is pretty involved. It is designed to show several different tools at work. Study it carefully, and observe you can work the problem one small part at a time to deal with the complexity.

$$\lim_{x\to 0}\frac{\arcsin x+\cos(3x)-e^x}{\arctan(3x)+x^2-\sin(3x)}\left(\tfrac{0}{0}\right) \overset{\text{L'H}}{=} \lim_{x\to 0}\frac{\dfrac{1}{\sqrt{1-x^2}}-3\sin(3x)-e^x}{\dfrac{1}{1+(3x)^2}\cdot(3)+2x-3\cos(3x)}\left(\tfrac{0}{0}\right)$$

$$\overset{\text{prep}}{=} \lim_{x\to 0}\frac{\left(1-x^2\right)^{-\frac{1}{2}}-3\sin(3x)-e^x}{3\left(1+9x^2\right)^{-1}+2x-3\cos(3x)}\left(\tfrac{0}{0}\right) \overset{\text{L'H}}{=} \lim_{x\to 0}\frac{-\dfrac{1}{2}\left(1-x^2\right)^{-\frac{3}{2}}\cdot(-2x)-9\cos(3x)-e^x}{-3\left(1+9x^2\right)^{-2}\cdot(18x)+2+9\sin(3x)}$$

$$\overset{\text{rewrite}}{=} \lim_{x\to 0}\frac{\cancelto{0}{\dfrac{x}{(1-x^2)^{\frac{3}{2}}}}-9\cancelto{1}{\cos(3x)}-\cancelto{1}{e^x}}{\cancelto{0}{\dfrac{-54x}{(1+9x^2)^2}}+2+9\cancelto{0}{\sin(3x)}} = \frac{-9-1}{2} = \frac{-10}{2} = \boxed{-5}$$

Example Evaluate $\displaystyle\lim_{x\to\infty}\left(1-\frac{2}{x^3}\right)^{x^3}$.

Solution This is in the indeterminate form 1^∞. We can deal with this by taking the logarithm:

$$\begin{aligned}\lim_{x\to\infty}\left(1-\frac{2}{x^3}\right)^{x^3} &\overset{1^\infty}{=} \lim_{x\to\infty} e^{\ln\left(\left(1-\frac{2}{x^3}\right)^{x^3}\right)}\\ &= e^{\lim\limits_{x\to\infty} x^3\ln\left(1-\frac{2}{x^3}\right)}\end{aligned}$$

Now focus on evaluating that new limit in the exponent.

$$\begin{aligned}\lim_{x\to\infty} x^3\ln\left(1-\frac{2}{x^3}\right) &= \lim_{x\to\infty}\frac{\ln\left(1-\frac{2}{x^3}\right)}{\frac{1}{x^3}}\\ &\overset{\left(\frac{0}{0}\right)^{\text{L'H}}}{=} \lim_{x\to\infty}\frac{\left(\frac{1}{1-\frac{2}{x^3}}\right)\left(\frac{6}{x^4}\right)}{-\frac{3}{x^4}}\\ &= \lim_{x\to\infty}\left(\frac{1}{1-\frac{2}{x^3}}\right)\left(\frac{6}{\cancel{x^4}}\right)\cdot\left(-\frac{\cancel{x^4}}{3}\right) = \lim_{x\to\infty}\left(\frac{1}{1-\cancelto{0}{\frac{2}{x^3}}}\right)(-2)\\ &= 1\cdot(-2) = -2\end{aligned}$$

And therefore the value of the original limit is e^{-2}.

Problems to hand in

Compute each of the following Limits. Simplify. *Justify every step.*

1. $\displaystyle\lim_{\theta\to\frac{\pi}{2}} \frac{1-\sin\theta}{1+\cos(2\theta)}$
2. $\displaystyle\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}$
3. $\displaystyle\lim_{x\to 0^+} \frac{\ln x}{x}$
4. $\displaystyle\lim_{x\to 0} \frac{e^{2x}-1-2x}{x^2}$
5. $\displaystyle\lim_{x\to 0} \frac{\sin x - x}{x^3}$
6. $\displaystyle\lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{x-\sin x}$
7. $\displaystyle\lim_{x\to 0} \frac{\arcsin(3x)}{\arctan(4x)}$
8. $\displaystyle\lim_{x\to 0} \frac{x-\arcsin x}{\arctan(2x)-2x}$
9. $\displaystyle\lim_{x\to 0} \frac{3xe^x - \arctan(3x)}{x+\ln(1-x)}$
10. $\displaystyle\lim_{x\to 0} \frac{\arcsin x + x^2 - x}{\cos x - \arctan(5x) - e^{-5x}}$
11. $\displaystyle\lim_{x\to\infty} x\sin\left(\frac{\pi}{x}\right)$
12. $\displaystyle\lim_{x\to\infty} x\ln\left(1-\frac{1}{x}\right)$
13. $\displaystyle\lim_{x\to 0^+} x\ln x$
14. $\displaystyle\lim_{x\to 0^+} \sqrt{x}\ln x$
15. $\displaystyle\lim_{x\to\infty} x^2e^{-x}$
16. $\displaystyle\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x$
17. $\displaystyle\lim_{x\to 0^+} (1+\ln(1-3x))^{\frac{1}{x}}$
18. $\displaystyle\lim_{x\to\infty} \left(1-\arctan\left(\frac{7}{x^4}\right)\right)^{x^4}$