Goal Practice L'Hôpital's rule; integrate by parts.
Reference: §6.8, 7.1

Examples to read first

Example Evaluate $\lim\limits_{x\to\infty} \left(e^{\frac{1}{x}} - \frac{4}{x}\right)^x$.

Solution Taking the logarithm and passing the limit to the exponent:

$$\lim_{x\to\infty} \left(e^{\frac{1}{x}} - \frac{4}{x}\right)^x \overset{1^\infty}{=} \lim_{x\to\infty} e^{\ln\left(\left(e^{\frac{1}{x}} - \frac{4}{x}\right)^x\right)} = e^{\lim\limits_{x\to\infty} \ln\left(\left(e^{\frac{1}{x}} - \frac{4}{x}\right)^x\right)}$$

Now, evaluating this limit in the exponent by itself:

$$\begin{aligned}
\lim_{x\to\infty} \ln\left(\left(e^{\frac{1}{x}} - \frac{4}{x}\right)^x\right) &= \lim_{x\to\infty} x\ln\left(e^{\frac{1}{x}} - \frac{4}{x}\right) \\
&\overset{\infty\cdot 0}{=} \lim_{x\to\infty} \frac{\ln\left(e^{\frac{1}{x}} - \frac{4}{x}\right)}{\frac{1}{x}} \\
&\overset{\left(\frac{0}{0}\right)^{\text{L'H}}}{=} \lim_{x\to\infty} \frac{\left(\frac{1}{e^{\frac{1}{x}} - \frac{4}{x}}\right) \cdot \left[e^{\frac{1}{x}}\left(-\frac{1}{x^2}\right) + \frac{4}{x^2}\right]}{-\frac{1}{x^2}} \\
&= \lim_{x\to\infty} \left(\frac{1}{e^{\frac{1}{x}} - \frac{4}{x}}\right) \cdot \left[e^{\frac{1}{x}}\left(-\frac{1}{\cancel{x^2}}\right) + \frac{4}{\cancel{x^2}}\right](-\cancel{x^2}) \\
&= \left(\frac{1}{e^0 - 0}\right)\left[e^0 - 4\right] \\
&= -3.
\end{aligned}$$

Therefore the original limit is $\boxed{e^{-3}}$

Example Evaluate $\int \arctan\left(\frac{1}{x}\right) dx$.

Solution We integrate by parts, with the following choices:

$u = \arctan\left(\frac{1}{x}\right)$	$dv = 1dx$
$du = \frac{1}{1+\left(\frac{1}{x}\right)^2} \cdot \left(-\frac{1}{x^2}\right) dx$	$v = x$
$du = -\frac{1}{x^2+1} dx$	$\leftarrow$ simplify

to obtain:

$$\begin{aligned} \int \arctan\left(\frac{1}{x}\right)\ dx &= x \arctan\left(\frac{1}{x}\right) - \left(- \int \frac{x}{x^2+1}\ dx\right) \\ &= x \arctan\left(\frac{1}{x}\right) + \frac{1}{2}\ln\left|x^2+1\right| + C, \end{aligned}$$

where in the last line we have performed the following substitution (written with w since u and v have been used already).

$$\begin{aligned} w &= x^2+1 \\ dw &= 2x\ dx \\ \tfrac{1}{2}dw &= x\ dx \end{aligned}$$

1 Problems to hand in

1. $\lim\limits_{x\to\infty} \frac{\ln\left(5+e^{3x}\right)}{x}$

2. $\lim\limits_{x\to\infty} \left(\frac{x}{x+1}\right)^x$

3. $\lim\limits_{x\to\infty} \left(e^{\frac{1}{x^6}} - \frac{6}{x^6}\right)^{x^6}$

Compute each of the following Integrals using Integration by Parts. Simplify.

4. $\int x\cos(5x)\ dx$

5. $\int_0^1 \arctan x \; dx$

6. $\int_0^5 \frac{x^2}{e^x} \; dx$

7. $\int (\ln x)^2 \; dx$

8. $\int_1^{\sqrt{3}} \arctan\left(\frac{1}{x}\right) \; dx$

9. $\int x \arctan x \; dx$

10. $\int \ln\left(x^2+7\right) \; dx$