MATH 19
FINAL EXAM PRACTICE
14 NOVEMBER 2014

Name : Solutions

Show all of your reasoning. You may use the back of each page for additional space or scratch work. You do not need to simplify your answers unless specifically instructed to do so.

You may use one page of notes (front and back). You do not need to submit it with the exam.

1	/25	**2**	/10	**3**	/10
4	/10	**5**	/10	**6**	/10
7	/10	**8**	/10	**9**	/10
10	/15	**11**	/15	**12**	/15
				Σ	/150

(1) **Short answer questions.** You do not need to show any work for the following questions.

(a) Evaluate $\sum_{n=2}^{\infty} \frac{18}{3^n}$ Answer: 3

$$= \frac{18}{3^2} + \frac{18}{3^3} + \cdots$$

$$= \frac{2}{1-1/3} = \frac{2}{2/3}$$

(b) Evaluate $\int_1^{\infty} \frac{1}{x^2}\, dx$ Answer: 1

$$= \left[-1/x\right]_1^{\infty}$$

(c) Determine the Taylor series of $f(x) = \sin(2x)$ around the center $x = 0$.

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot (2x)^{2n+1}$$

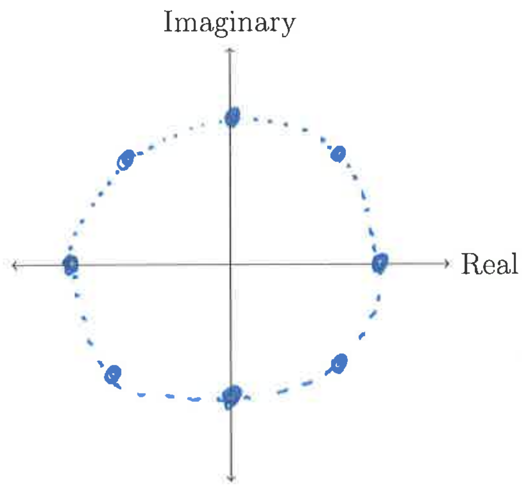
Answer: $\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 2^{2n+1}}{(2n+1)!} \cdot x^{2n+1}$

(d) Determine the Taylor series of $f(x) = xe^x$ around the center $x = 0$.

$$x \cdot \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

Answer: $\sum_{n=0}^{\infty} \frac{1}{n!} x^{n+1}$ $\left(\text{or } \sum_{n=1}^{\infty} \frac{1}{(n-1)!} x^n\right)$

(e) Sketch on the following axes the locations of the complex solutions to the equation $z^8 = 1$.



(f) Find a function $f(t)$ satisfying the following three properties.

$$\int_{-\pi}^{\pi} f(t) = 8\pi \quad \Leftrightarrow \quad a_0 = 4$$

$$\int_{-\pi}^{\pi} f(t)\sin t = 3\pi \quad \Leftrightarrow \quad b_1 = 3$$

$$\int_{-\pi}^{\pi} f(t)\cos(2t) = 4\pi \quad \Leftrightarrow \quad a_2 = 4$$

Answer: $4 + 3\sin t + 4\cos(2t)$

(2) Solve the following initial value problem.

$$\begin{aligned} y''(t) + 6y'(t) + 8y(t) &= 24 \\ y(0) &= 0 \\ y'(0) &= 14 \end{aligned}$$

$y'' + 6y' + 8(y-3) = 0$

let $u = y-3$. So $u'' + 6u' + 8u = 0$.

char. ~~poly~~ eqn.: $\lambda^2 + 6\lambda + 8 = 0$

$(\lambda+2)(\lambda+4) = 0$

$\lambda = -2$ or $\lambda = -4$

$\Rightarrow \quad u(t) = C \cdot e^{-2t} + D \cdot e^{-4t}$

& $y(t) = C \cdot e^{-2t} + D \cdot e^{-4t} + 3.$

$y'(t) = -2Ce^{-2t} - 4De^{-4t}$

$0 = y(0) = C + D + 3$

& $14 = y'(0) = -2C - 4D$

$D = -C - 3$

$14 = -2C - 4(-C-3)$

$= -2C + 4C + 12 = 2C + 12$

$\Rightarrow C = 1$ & $D = -4$

$\Rightarrow \quad \boxed{y(t) = e^{-2t} - 4e^{-4t} + 3}$

(3) Evaluate $\int_0^\infty x^2 e^{-x/2}\, dx$.

$u = x^2 \qquad dv = e^{-x/2}dx$

$du = 2x\,dx \qquad v = -2e^{-x/2}$

$$= \left[-2x^2 e^{-x/2}\right]_0^\infty - \int_0^\infty (-2)\, e^{-x/2} \cdot 2x\,dx$$

$$= \underbrace{\left[\lim_{x\to\infty} \frac{-2x^2}{e^{x/2}} - 0\right]} + 4\int_0^\infty x \cdot e^{-x/2} dx$$

$u = x \qquad dv = e^{-x/2}dx$

$du = dx \qquad v = -2e^{-x/2}$

0, since

$\lim \frac{x^2}{e^{x/2}}$

$= \lim \frac{2x}{\frac{1}{2}e^{x/2}}$

$= \lim \frac{2}{\frac{1}{4}e^{x/2}}$

$= 0$ by repeated l'hôpital

$$= 4\left[-2x \cdot e^{-x/2}\right]_0^\infty - 4\int_0^\infty (-2)\, e^{-x/2} dx$$

$$= \lim_{x\to\infty}\left(\frac{-8x}{e^{x/2}}\right) + 8\int_0^\infty e^{-x/2} dx$$

$$= 0 + 8 \cdot \left[-2e^{-x/2}\right]_0^\infty$$

e.g. by l'hôpital.

$$= 8 \cdot [0 - (-2) \cdot 1]$$

$$= \boxed{16}$$

(4) Evaluate $\int_0^{\pi/4} \sec^4 x \tan x \, dx$.

$$= \int_0^{\pi/4} \sec^2 x \cdot (1+\tan^2 x) \cdot \tan x \, dx$$

$u = \tan x$
$du = \sec^2 x$

$$= \int_0^1 (1+u^2) \cdot u \, du \quad = \int_0^1 (u+u^3) du$$

$$= \left[\frac{1}{2}u^2 + \frac{1}{4}u^4\right]_0^1$$

$$= \frac{1}{2} + \frac{1}{4}$$

$$= \boxed{3/4}$$

(5) Determine whether each series converges or diverges. Be specific about which tests or facts you are using.

(a) $\sum_{n=1}^{\infty} \frac{n^2+n}{n^3-n}$

comparison:

$$0 \le \frac{n^2}{n^3} \le \frac{n^2+n}{n^3-n}$$

and $\sum \frac{n^2}{n^3} = \sum \frac{1}{n}$ diverges by integral test

$\Rightarrow$ this series $\boxed{\text{diverges}}$.

(b) $\sum_{n=1}^{\infty} \frac{\ln n}{2^n}$

ratio

$$L = \lim_{n\to\infty} \frac{\ln(n+1)}{\ln(n)} \cdot \frac{2^n}{2^{n+1}}$$

$$= \frac{1}{2} \cdot \lim_{n\to\infty} \frac{\ln(n+1)}{\ln(n)} = \frac{1}{2} \lim_{n\to\infty} \frac{n}{n+1}$$

$$= \frac{1}{2}.$$

Since $1/2 < 1$, the series $\boxed{\text{converges.}}$

(c) $\sum_{n=1}^{\infty} \frac{1}{n \cdot [1+(\ln n)^2]}$

integral

$$\int_1^{\infty} \frac{1}{x \cdot (1+(\ln x)^2)}\,dx \qquad u = \ln x \qquad du = \frac{1}{x}\,dx$$

$$= \int_0^{\infty} \frac{1}{1+u^2}\,du = \left[\tan^{-1} u\right]_0^{\infty}$$

$$= \lim_{u\to\infty} \tan^{-1}(u) = \pi/2$$

integral converges $\Rightarrow$ series $\boxed{\text{converges.}}$

(6) Convert the following equation to rectangular form (an equation in variables x and y).

$$r = \frac{1}{3 + \cos\theta + 2\sin\theta}$$

$3r + r \cdot \cos\theta + 2r\sin\theta = 1$

$3r = 1 - x - 2y$

~~3r~~

$9r^2 = (1-x-2y)^2$

$\boxed{9x^2 + 9y^2 = (1-x-2y)^2}$

(7) Convert the (finite) complex Fourier series below to a (finite) real Fourier series.

$(3+4i)e^{-3it} + (5-12i)e^{-it} + (5+12i)e^{it} + (3-4i)e^{3it}$

$$= 3\cdot\left(e^{-3it} + e^{3it}\right) + 4i\left(e^{-3it} - e^{3it}\right)$$

$$+ 5\left(e^{-it} + e^{it}\right) - 12i\left(e^{-it} - e^{it}\right)$$

$$= 3\cdot 2\cos(3t) + 4\cdot 2\sin(3t)$$

$$+ 5\cdot 2\cos t - 12\cdot 2\cdot \sin t$$

$$= \boxed{10\cos t - 24\sin t + 6\cos(3t) + 8\sin 3t}$$

(8) Evaluate each of the following sums.

(a) $\sum_{n=0}^{\infty} \frac{(-1)^n}{5^n \cdot n!}$ $= \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \left(-\frac{1}{5}\right)^n$

$= \boxed{e^{-1/5}}$

(b) $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)! \cdot 5^n}$ $= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!\,(\sqrt{5})^{2n}}$

$= \sqrt{5} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot \left(\frac{1}{\sqrt{5}}\right)^{2n+1}$

$= \boxed{\sqrt{5} \cdot \sin\left(\frac{1}{\sqrt{5}}\right)}$

(c) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n \cdot 5^n}$ $= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cdot \left(\frac{1}{5}\right)^n$

$= \ln\left(1+\frac{1}{5}\right) = \boxed{\ln(6/5)}$

or $\ln 6 - \ln 5$.

(9) Find the fourth order Taylor approximation of the function

$$f(x) = e^{-x^2} \sin x$$

around the center $x = 0$.

$$f(x) = \left(1 + (-x^2) + \frac{1}{2}(-x^2)^2 + \cdots\right) \cdot \left(x - \frac{1}{6}x^3 + \cdots\right)$$

$$= \left(1 - x^2 + \frac{1}{2}x^4 - \cdots\right)\left(x - \frac{1}{6}x^3 + \cdots\right)$$

$$= 1 \cdot x + 1 \cdot \left(-\frac{1}{6}\right)x^3 - x^2 \cdot x + \cdots$$ (all other terms have degree ≥ 5)

$$= x + \left(-\frac{1}{6} - 1\right)x^3 + \cdots$$

so fourth order approx. is

$$P_4(x) = \boxed{x - \frac{7}{6}x^3}$$

(10) Define a path by the following position function.

$$\vec{r}(t) = \left(\frac{1}{2}t^2, 5, \frac{1}{3}t^3\right)$$

Determine the arc-length of this path from $t = 0$ to $t = 1$.

velocity $\vec{v}(t) = (t, 0, t^2)$

speed $|\vec{v}(t)| = \sqrt{t^2 + t^4}$

$\Rightarrow$ arclength $= \int_0^1 |\vec{v}(t)| dt$

$= \int_0^1 \sqrt{t^2+t^4}\, dt$

$= \int_0^1 t \cdot \sqrt{1+t^2}\, dt$ $\quad$ $t = \tan\vartheta$ $\quad$ $dt = \sec^2\vartheta$ $\quad$ $\sqrt{1+t^2} = \sec\vartheta$ $\quad$ $\vartheta = \tan^{-1}(t)$

$= \int_0^{\pi/4} \tan\vartheta \cdot \sec\vartheta \cdot \sec^2\vartheta\, d\vartheta$

$u = \sec\vartheta$ $\quad$ $du = \sec\vartheta \tan\vartheta\, d\vartheta$

$= \int_1^{\sqrt{2}} u^2 du = \left[\frac{1}{3}u^3\right]_1^2$

$= \boxed{\frac{1}{3}(2\sqrt{2} - 1)}$

(11) Let $f(t)$ be the solution to the following initial value problem (you do not need to find an explicit formula for $f(t)$).

$$\begin{aligned} f''(t) + f'(t) + f(t) &= e^{-t^2} \\ f(0) &= 1 \\ f'(0) &= 1 \end{aligned}$$

(a) Determine the values of $f''(0)$, $f'''(0)$, and $f^{(4)}(0)$.

$f''(0) = e^0 - f'(0) - f(0) = 1 - 1 - 1 = \underline{\underline{-1}}$

$f'''(t) + f''(t) + f'(t) = -2t \cdot e^{-t^2}$

$\Rightarrow f'''(0) = -2 \cdot 0 \cdot e^0 - f''(0) - f'(0) = 0 - (-1) - 1 = \underline{\underline{0}}$

$f^{(4)}(t) + f'''(t) + f''(t) = -2e^{-t^2} - 2t \cdot (-2t) \cdot e^{-t^2}$

$\Rightarrow f^{(4)}(0) = -2 \cdot e^0 - 2 \cdot 0 \cdot (-2 \cdot 0) \cdot e^0 - f'''(0) - f''(0)$

$= -2 - 0 - 0 - (-1) = \underline{\underline{-1}}$

(b) Find the fourth order Taylor approximation of $f(x)$ around the center $x = 0$.

$P_4(x) = \boxed{1 + x - \frac{1}{2}x^2 - \frac{1}{24}x^4}$

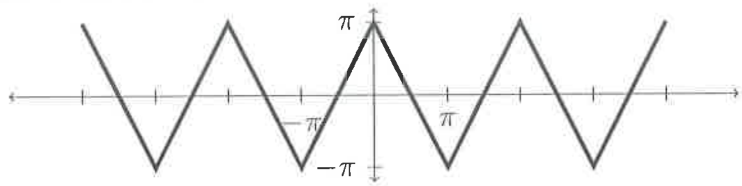
(c) Give an approximation of the value $f(1)$.

$f(1) \approx 1 + 1 - \frac{1}{2} - \frac{1}{24}$

$= 2 - \frac{13}{24}$

$= 1\frac{11}{24}$ or $\boxed{\frac{35}{24}}$

(12) Find the (2π-periodic) real Fourier series of the function whose graph is shown below.



$$f(x) = \begin{cases} \pi + 2x & \text{for } -\pi \le x \le 0 \\ \pi - 2x & \text{for } 0 \le x \le \pi \end{cases}$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0}(\pi+2x)\,dx + \int_0^{\pi}(\pi-2x)\,dx\right]$$

$$= \frac{1}{2\pi}\cdot\left[\pi\cdot x + x^2\right]_{-\pi}^{0} + \frac{1}{2\pi}\left[\pi x - x^2\right]_0^{\pi} = \frac{1}{2\pi}\left(+\pi^2-\pi^2\right) + \frac{1}{2\pi}\left(\pi^2-\pi^2\right)$$

$$= 0.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\cos(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0}(\pi+2x)\cos(nx)\,dx + \frac{1}{\pi}\int_0^{\pi}(\pi-2x)\cos(nx)\,dx$$

$u = \pi + 2x$, $du = 2\,dx$, $dv = \cos(nx)\,dx$, $v = \frac{1}{n}\sin(nx)$; $u = \pi - 2x$, $du = -2dx$, $dv = \cos(nx)$, $v = \frac{1}{n}\sin(nx)$

$$= \frac{1}{\pi}\left[\frac{1}{n}\cdot(\pi+2x)\sin(nx)\right]_{-\pi}^{0} - \frac{1}{\pi}\int_{-\pi}^{0}\frac{1}{n}\sin(nx)\cdot 2dx + \frac{1}{\pi}\left[\left(\frac{1}{n}\right)\cdot(\pi-2x)\sin(nx)\right]_0^{\pi} - \frac{1}{\pi}\int_0^{\pi}\left(-\frac{2}{n}\right)\sin(nx)dx$$

$$= 0 - \frac{2}{n\pi}\int_{-\pi}^{0}\sin(nx)\,dx + 0 + \frac{2}{n\pi}\int_0^{\pi}\sin(nx)\,dx$$

$$= -\frac{2}{n\pi}\cdot\left[-\frac{1}{n}\cos(nx)\right]_{-\pi}^{0} + \frac{2}{n\pi}\cdot\left[-\frac{1}{n}\cos(nx)\right]_0^{\pi}$$

$$= \frac{2}{n\pi}\cdot\left[\frac{1}{n}\cos 0 - \frac{1}{n}\cos(n\pi)\right] + \frac{2}{n\pi}\cdot\left[-\frac{1}{n}\cos(n\pi) + \frac{1}{n}\cos 0\right]$$

$$= \frac{2}{n^2\pi}\cdot\left(1-(-1)^n\right) + \frac{2}{n^2\pi}\cdot\left(-(-1)^n+1\right)$$

$$= \frac{4}{n^2\pi}\cdot\left(1-(-1)^n\right) = \begin{cases} 8/n^2\pi & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

(continued on next page)

Additional space for work

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\sin(nx)dx = \frac{1}{\pi}\int_{-\pi}^{0}(\pi+2x)\sin(nx)dx + \frac{1}{\pi}\int_{0}^{\pi}(\pi-2x)\sin(nx)dx$$

$u=\pi+2x$ $\quad dv=\sin(nx)dx$

$du=2dx$ $\quad v=\frac{-1}{n}\cos(nx)$

$u=\pi-2x$ $\quad dv=\sin(nx)dx$

$du=-2dx$ $\quad v=-\frac{1}{n}\cos(nx)$

$$= \frac{1}{\pi}\left[-\frac{1}{n}\cdot(\pi+2x)\cos(nx)\right]_{-\pi}^{0} - \frac{1}{\pi}\int_{-\pi}^{0}\left(-\frac{2}{n}\right)\cos(nx)dx$$

$$+ \frac{1}{\pi}\left[-\frac{1}{n}(\pi-2x)\cos(nx)\right]_{0}^{\pi} - \frac{1}{\pi}\int_{0}^{\pi}\left(-\frac{1}{n}\right)\cos(nx)\cdot(-2)dx$$

$$= \frac{1}{\pi}\cdot\left[-\frac{1}{n}\cdot\pi\cdot\cos 0 + \frac{1}{n}\cdot(-\pi)\cdot\cos(-n\pi)\right] + \frac{2}{n\pi}\int_{-\pi}^{0}\cos(nx)dx$$

$$+\frac{1}{\pi}\left[-\frac{1}{n}\cdot(-\pi)\cdot\cos(n\pi) + \frac{1}{n}\cdot\pi\cdot\cos 0\right] - \frac{2}{n\pi}\int_{0}^{\pi}\cos(nx)dx$$

$$= \cancel{-\frac{1}{n}} - \cancel{\frac{1}{n}\cdot(-1)^n} + \frac{2}{n\pi}\cdot\left[\frac{1}{n}\sin(nx)\right]_{-\pi}^{0}$$

$$+\cancel{\frac{1}{n}\cdot(-1)^n} + \cancel{\frac{1}{n}} - \frac{2}{n\pi}\left[\frac{1}{n}\sin(nx)\right]_{0}^{\pi}$$

$$= 0.$$

So the Fourier series is

$$\boxed{\sum_{n=1}^{\infty}\frac{4}{\pi n^2}\cdot\left(1-(-1)^n\right)\cos(nx)}$$

or, equivalently, $\boxed{\displaystyle\sum_{n=0}^{\infty}\frac{8}{\pi(2n+1)^2}\cdot\cos\left((2n+1)x\right)}$

Note: you can economize this computation in two places, since $f(x)$ is an even function: $f(x)\cdot\cos(nx)$ is odd, so $a_n = \frac{1}{\pi}\int_0^{\pi} f(x)\cdot\cos(nx)dx$, and $f(x)\cdot\sin(nx)$ is even, so b_n is 0 automatically.