MATH 19
FINAL EXAM
14 DECEMBER 2014

Name : Solutions

Show all of your reasoning. You may use the back of each page for additional space or scratch work. You do not need to simplify your answers unless specifically instructed to do so.

You may use one page of notes (front and back). You do not need to submit it with the exam. No other calculators or aids are permitted.

1	/25	**2**	/10	**3**	/10
4	/10	**5**	/10	**6**	/10
7	/10	**8**	/10	**9**	/10
10	/15	**11**	/15	**12**	/15
				Σ	/150

(1) **Short answer questions.** You do not need to show any work for the following questions.

(a) Evaluate $\int_0^\infty e^{-x/3}\,dx$. Answer: 3

$$\left[-3e^{-x/3}\right]_0^\infty$$

(b) Determine whether each series converges or diverges.

$\sum_{n=1}^\infty 1$	$\sum_{n=1}^\infty \frac{1}{\sqrt{n}}$	$\sum_{n=1}^\infty \frac{1}{n}$	$\sum_{n=1}^\infty \frac{1}{n^2}$
☐ converges ☑ diverges	☐ converges ☑ diverges	☐ converges ☑ diverges	☑ converges ☐ diverges
$\sum_{n=1}^\infty (-1)^{n-1}$	$\sum_{n=1}^\infty \frac{(-1)^{n-1}}{\sqrt{n}}$	$\sum_{n=1}^\infty \frac{(-1)^{n-1}}{n}$	$\sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^2}$
☐ converges ☑ diverges	☑ converges ☐ diverges	☑ converges ☐ diverges	☑ converges ☐ diverges

(c) Find the angle between the following two vectors. You do not need to simplify your answer.

$$\vec{v} = (1,1,1)$$
$$\vec{w} = (1,-1,1)$$

$\vec{v}\cdot\vec{w} = 1-1+1=1$

$\vec{v}\cdot\vec{v} = 3$

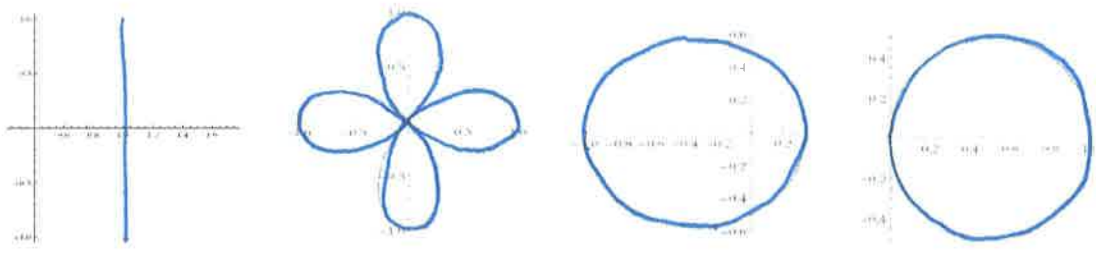
$\vec{w}\cdot\vec{w} = 3$

$\cos\theta = \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\cdot\|\vec{w}\|}$

Answer: $\cos^{-1}\left(\frac{1}{3}\right)$

(d) Evaluate $\sum_{n=0}^\infty \frac{2^n}{n!}$. Answer: e^2

(e) Identify which polar equation describes each graph (circle the appropriate letter).



A B C D   A B C D   A B C D   A B C D

$$
\begin{aligned}
A: &\quad r = \cos\theta \\
B: &\quad r = \cos(2\theta) \\
C: &\quad r = \frac{1}{\cos\theta} \\
D: &\quad r = \frac{1}{2+\cos\theta}
\end{aligned}
$$

(f) Determine the three complex Fourier coefficients c_{-1}, c_0, and c_1 of the following function.

$$f(x) = 2\cos x - 6\sin x \quad = \quad e^{-ix} + e^{ix} - 3i e^{-ix} + 3i e^{ix}$$

$c_{-1} =$ $1-3i$ $c_0 =$ 0 $c_1 =$ $1+3i$

(g) Determine the radius of convergence of the following series.

$$\sum_{n=0}^{\infty} 2^n x^n$$

$L = \lim \left| \frac{2^{n+1}x^{n+1}}{2^n x^n} \right|$

$= 2 \cdot |x|.$

Answer: $1/2$

(2) Solve the following initial value problem.

$$\begin{aligned} \frac{dy}{dx} &= y \cdot \cos(2x) \\ y(0) &= 2 \end{aligned}$$

$$\frac{1}{y}\,dy = \cos(2x)\,dx$$

$$\int \frac{1}{y}\,dy = \int \cos(2x)\,dx$$

$$\ln|y| = \tfrac{1}{2}\sin(2x) + C$$

$$|y| = e^{\frac{1}{2}\sin(2x)+C}$$

$$y = \pm e^{C} \cdot e^{\sin(2x)/2}$$

$$y(0) = 2 = \pm e^{C} \cdot e^{0}$$

$$\Rightarrow \pm e^{C} = 2.$$

$$\boxed{y(x) = 2e^{\sin(2x)/2}}$$

(3): (a) Find the quadratic approximation of the function $f(x) = (1+x)^{100}$ around the center $x = 0$.
(b) Use this to approximate the value of 1.001^{100}.

a)

$$f(x) = (1+x)^{100}$$
$$f'(x) = 100 \cdot (1+x)^{99}$$
$$f''(x) = 9900 \cdot (1+x)^{98}$$

$$f(0) = 1$$
$$f'(0) = 100$$
$$f''(0) = 9900$$

so

$$P_2(x) = 1 + 100x + \frac{9900}{2} \cdot x^2$$
$$= \boxed{1 + 100x + 4950x^2}$$

b)

$$1.001^{100} \approx P_2(0.001)$$
$$= 1 + 0.1 + 0.004950$$
$$= \boxed{1.10495}$$

(4) Let $\vec{f}$ and $\vec{v}$ be the following vectors.

$$\begin{aligned} \vec{f} &= (7,8,19) \\ \vec{v} &= (1,0,1) \end{aligned}$$

Find two vectors $\vec{f_1}, \vec{f_2}$ such that $\vec{f_1}$ is parallel to $\vec{v}$, $\vec{f_2}$ is perpendicular to $\vec{v}$, and $\vec{f} = \vec{f_1} + \vec{f_2}$.

$\vec{f_1} = \lambda \cdot \vec{v}$ for some λ (since $\vec{f_1} \parallel \vec{v}$)

$\vec{f_2} \cdot \vec{v} = 0$ (since $\vec{f_2} \perp \vec{v}$)

$\vec{f_2} = \vec{f} - \vec{f_1}$

$\Rightarrow$ $0 = \vec{f_2} \cdot \vec{v} = \vec{f} \cdot \vec{v} - \lambda (\vec{v} \cdot \vec{v})$

$0 = (7 \cdot 1 + 8 \cdot 0 + 19 \cdot 1) - \lambda \cdot (1^2 + 0^2 + 1^2)$

$0 = 26 - 2\lambda$

$\Rightarrow$ $\lambda = 13.$

So $\vec{f_1} = 13 \cdot \vec{v} = (13,0,13)$

& $\vec{f_2} = \vec{f} - \vec{f_1} = (7,8,19) - (13,0,13)$

$= (-6,8,6)$

$$\boxed{\begin{aligned} \vec{f_1} &= (13,0,13) \\ \vec{f_2} &= (-6,8,6) \end{aligned}}$$

(5) Evaluate the following sum where it converges (as a function of x).

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k \cdot x^{2k}}$$

Recall that $\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \cdot x^k$.

Therefore

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k x^{2k}} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \cdot \left(\frac{1}{x^2}\right)^k$$

$$= \boxed{\ln\left(1+\frac{1}{x^2}\right)}$$ (when the series converges).

(6) Find complex numbers z, w satisfying the following two equations.

$$\begin{aligned}(1+2i)w + (3-4i)z &= 30\\ 2w + z &= 5\end{aligned}$$

$z = (5-2w)$

$(1+2i)w + (3-4i)(5-2w) = 30$

$(1+2i)w + (15-20i) + (-6+8i)w = 30$

$(-5+10i)w = 15+20i$

$$w = \frac{15+20i}{-5+10i}$$

$$= \frac{3+4i}{-1+2i} \cdot \frac{-1-2i}{-1-2i}$$

$$= \frac{-3+8-6i-4i}{5} = \frac{5-10i}{5}$$

$= 1-2i.$

and $z = 5-2w = 5-2(1-2i) = 3+4i.$

$$\boxed{\begin{aligned} w &= 1-2i\\ z &= 3+4i\end{aligned}}$$

(7) Find a series of rational numbers whose sum converges to the value of the following integral.

$$\int_0^1 x \cdot e^{-x^3}\, dx$$

$$\int_0^1 x \cdot \sum_{n=0}^{\infty} \frac{1}{n!} (-x^3)^n \, dx$$

$$= \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{3n+1} \, dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left[\frac{1}{3n+2} \cdot x^{3n+2} \right]_0^1$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{n! \cdot (3n+2)}}$$

(8) Let $f(t)$ be a function defined by the following Fourier series.

$$f(t) = \sum_{n=1}^{\infty} \frac{1}{2^n} \sin(nt)$$

(a) Show that this series converges for all values of t by using appropriate convergence tests.

comparison test:

$0 \leq \left| \frac{1}{2^n} \sin(nt) \right| \leq \frac{1}{2^n}$, and $\sum \frac{1}{2^n}$ converges since it is geometric of ratio $\frac{1}{2} < 1$.

$\Rightarrow \sum_{n=1}^{\infty} \left| \frac{1}{2^n} \sin(nt) \right|$ converges

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \sin(nt)$ converges (absolutely).

(b) Determine the values of the following integrals. Correct answers will receive full credit even if no work is shown.

$$\int_{-\pi}^{\pi} f(t)\, dt = \underline{\quad 0 \quad} \qquad (\text{this is } 2\pi \cdot a_0)$$

$$\int_{-\pi}^{\pi} f(t) \sin(3t)\, dt = \underline{\quad \pi/8 \quad} \qquad (\text{this is } \pi \cdot b_3)$$

$$\int_{-\pi}^{\pi} f(t) \cos(7t)\, dt = \underline{\quad 0 \quad} \qquad (\text{this is } \pi \cdot a_7)$$

(the Fourier coefficients are

$a_0 = 0$

$a_n = 0 \qquad b_n = \frac{1}{2^n}$).

(9) Evaluate $\int_0^2 x^2\sqrt{4-x^2}\,dx$.

$$x = 2\sin\vartheta \qquad dx = 2\cos\vartheta\,d\vartheta$$

$$\sqrt{4-x^2} = 2\cos\vartheta \qquad \vartheta = \sin^{-1}\left(\frac{x}{2}\right)$$

$$= \int_0^{\pi/2} 4\sin^2\vartheta \cdot 2\cos\vartheta \cdot 2\cos\vartheta\,d\vartheta$$

$$= \int_0^{\pi/2} 16\sin^2\vartheta\cos^2\vartheta\,d\vartheta$$

$$= \int_0^{\pi/2} 16\cdot\tfrac{1}{2}(1-\cos 2\vartheta)\cdot\tfrac{1}{2}(1+\cos 2\vartheta)\,d\vartheta$$

$$= \int_0^{\pi/2} 4\cdot(1-\cos^2 2\vartheta)\,d\vartheta$$

$$= \int_0^{\pi/2} 4\cdot\left(1-\tfrac{1}{2}(1+\cos 4\vartheta)\right)d\vartheta$$

$$= \int_0^{\pi/2} 4\cdot\left(\tfrac{1}{2}-\tfrac{1}{2}\cos 4\vartheta\right)d\vartheta$$

$$= \int_0^{\pi/2} (2-2\cos 4\vartheta)\,d\vartheta$$

$$= \left[2\vartheta - \tfrac{1}{2}\sin 4\vartheta\right]_0^{\pi/2}$$

$$= 2\cdot\frac{\pi}{2} - \tfrac{1}{2}\cdot 0 - 2\cdot 0 + \tfrac{1}{2}\cdot 0$$

$$= \boxed{\pi}$$

(10) An airplane takes off from the point $(2, 0, 0)$ at time $t = 0$ and travels to the point $(-2, 8, 4)$ at constant speed over the course of 40 seconds. An observer stands at the point $(0, 2, 0)$ watching the plane's flight.

(a) Determine the position $\vec{r}(t)$ of the airplane at time t.

$$\text{Velocity} = \frac{\text{displacement}}{\text{time}} = \frac{(-2,8,4)-(2,0,0)}{40}$$

$$= \left(-\tfrac{1}{10}, \tfrac{1}{5}, \tfrac{1}{10}\right)$$

starting point $= (2,0,0)$

$$\Rightarrow \quad \boxed{\vec{r}(t) = (2,0,0) + t\cdot\left(-\tfrac{1}{10}, \tfrac{1}{5}, \tfrac{1}{10}\right)}$$

(b) What is the location of the airplane at the moment that it is closest to the observer?

Alt. sol'n: minimal dist. when $\vec{r}(t)-(0,2,0)$ is perpendicular to $\vec{v} = \left(-\frac{1}{10}, \frac{1}{5}, \frac{1}{10}\right)$. So write $0 = (\vec{r}(t)-(0,2,0))\cdot\vec{v}$ and solve to get $t=10$.

$$\text{distance to observer} = \|\vec{r}(t) - (0,2,0)\|$$

$$= \left\|\left(2-\tfrac{1}{10}t, -2+\tfrac{1}{5}t, \tfrac{1}{10}t\right)\right\|$$

$$= \sqrt{\left(2-\tfrac{1}{10}t\right)^2 + \left(-2+\tfrac{1}{5}t\right)^2 + \left(\tfrac{1}{10}t\right)^2}$$

$$= \sqrt{4-\tfrac{2}{5}t+\tfrac{1}{100}t^2+4-\tfrac{4}{5}t+\tfrac{1}{25}t^2+\tfrac{1}{100}t^2}$$

$$= \sqrt{\tfrac{3}{50}t^2-\tfrac{6}{5}t+8}$$

This is minimal when $\frac{d}{dt}\left(\frac{3}{50}t^2-\frac{6}{5}t+8\right)=0$, i.e. $\frac{6}{50}t = \frac{6}{5}$, i.e. $\underline{t=10}$.

At this time,

$$\vec{r}(t) = \vec{r}(10) = (2,0,0) + (-1, 2, 1)$$

$$= \boxed{(1,2,1)}$$

(11) Find a vector function $\vec{r}(t)$ satisfying the following conditions (here $\vec{v}(t)$ and $\vec{a}(t)$ denotes the velocity and acceleration, respectively).

$$\begin{aligned} \vec{a}(t) &= -2\vec{v}(t) - 2\vec{r}(t) \\ \vec{r}(0) &= (0,1,-1) \\ \vec{v}(0) &= (1,0,1) \end{aligned}$$

Each coordinate individually satisfies the diff Eq

$$f''(t) = -2f'(t) - 2f(t)$$

i.e. $f'' + 2f + 2 = 0$.

This has char. eqn. $\lambda^2 + 2\lambda + 2 = 0$,

w/ sol'ns $\lambda = -1 \pm i$

$\Rightarrow$ a complex sol'n is $e^{(-1+i)t} = e^{-t} \cdot \cos t + i e^{-t} \sin t$.

$\Rightarrow$ the gen'l solution is

$$f(t) = C \cdot e^{-t} \cos t + D \cdot e^{-t} \sin t.$$

So there are constant vectors $\vec{C}, \vec{D}$ such that

$$\vec{r}(t) = \vec{C} \cdot e^{-t} \cos t + \vec{D} \cdot e^{-t} \sin t$$
$$\vec{v}(t) = \vec{C} \cdot (-e^{-t} \cos t - e^{-t} \sin t) + \vec{D} \cdot (-e^{-t} \sin t + e^{-t} \cos t)$$

$$\Rightarrow \quad \begin{aligned} \vec{r}(0) &= \vec{C} \\ \vec{v}(0) &= -\vec{C} + \vec{D}. \end{aligned}$$

Thus $\vec{C} = (0,1,-1)$ & $\vec{D} = (1,0,1) + \vec{C} = (1,1,0)$.

so

$$\vec{r}(t) = (0,1,-1) \cdot e^{-t} \cos t + (1,1,0) \cdot e^{-t} \sin t$$
$$= \boxed{\left(e^{-t} \sin t,\ e^{-t}(\cos t + \sin t),\ -e^{-t} \cos t\right)}$$

(12) Let $f(t)$ be the steady-state solution (the unique solution that is 2π-periodic) to the following differential equation.

$$f''(t) + 2f'(t) + f(t) = 2\sin(2t) - 4\cos(3t)$$

(a) Determine the complex Fourier coefficients of the function $V(t) = 2\sin(2t) - 4\cos(3t)$.

$$V(t) = i\cdot e^{-2it} - i\,e^{2it} - 2e^{-3it} - 2e^{3it}$$

so

$$C_{-3} = -2$$
$$C_{-2} = i$$
$$C_{2} = -i$$
$$C_{3} = -2$$

& all other $C_n = 0$

(b) Determine the complex Fourier coefficients of $f(t)$.

$$C_n(f) = \frac{1}{(in)^2 + 2in + 1}\cdot C_n(V) = \frac{1}{1-n^2+2in}\cdot C_n(V).$$

Therefore

$$C_{-3} = \frac{1}{-8-6i}\cdot(-2) = \frac{2(8-6i)}{(8+6i)(8-6i)} = \frac{16-12i}{100} = \frac{4-3i}{25}$$

$$C_{-2} = \frac{1}{-3-4i}\cdot i = \frac{-i(3-4i)}{(3+4i)(3-4i)} = \frac{-4-3i}{25}$$

$$C_{2} = \frac{1}{-3+4i}\cdot(-i) = \frac{-i(-3-4i)}{(-3+4i)(-3-4i)} = \frac{-4+3i}{25}$$

$$C_{3} = \frac{1}{-8+6i}\cdot(-2) = \frac{-2\cdot(-8-6i)}{(-8+6i)(-8-6i)} = \frac{16+12i}{100} = \frac{4+3i}{25}$$

(c) Write $f(t)$ as a real Fourier series (in terms of sines and cosines). Simplify your answer enough that there are no imaginary numbers in the expression.

$$f(t) = \frac{1}{25}\left[(4-3i)e^{-3it} + (-4-3i)\cdot e^{-2it} + (-4+3i)\,e^{2it} + (4+3i)e^{3it}\right]$$

$$= \frac{1}{25}\cdot\left[4(e^{-3it}+e^{3it}) - 3i(e^{-3it}-e^{3it}) - 4(e^{-2it}+e^{2it}) - 3i(e^{-2it}-e^{2it})\right]$$

$$= \frac{1}{25}\cdot\left[8\cos 3t - 6\sin 3t - 8\cos 2t - 6\sin 2t\right]$$

$$= \boxed{-\frac{8}{25}\cos 2t - \frac{6}{25}\sin 2t + \frac{8}{25}\cos 3t - \frac{6}{25}\sin 3t}$$

Additional space for work