MATH 19
MIDTERM 1
10 OCTOBER 2014

Name : Solutions

Show all of your reasoning. You may use the back of each page for additional space or scratch work. You do not need to simplify your answers unless specifically instructed to do so. Each question is worth 10 points.

You may use one page of notes (front and back), which you must submit with the exam. No calculators or other aids are permitted.

1		3	
2		4	
		Total	

(1) The compressed gas in a piston exerts force $F(x) = kx^{-\gamma}$ on the piston when the piston is at position x, where k and γ are constants. Suppose that when $x = 1$ meter this force is 24 Newtons, and when $x = 4$ this force is 3 Newtons. Determine the work done by the gas when it expands and pushes the piston from $x = 1$ to $x = 9$.

$\left.\begin{array}{l} F(1)=24 \\ F(4)=3 \end{array}\right\}$ Therefore $k\cdot 1^{-\gamma}=24 \Rightarrow \underline{k=24}$

and $k\cdot 4^{-\gamma}=3 \Rightarrow 4^{-\gamma}=\frac{3}{24}=\frac{1}{8}$

$\Rightarrow \underline{\gamma=3/2}$.

So the work is

$$\int_1^9 F(x)\,dx = \int_1^9 24x^{-3/2}\,dx$$

$$= \left[-48x^{-1/2}\right]_1^9 = 48\cdot\left(1^{-1/2}-9^{-1/2}\right)$$

$$= 48\cdot\left(1-\tfrac{1}{3}\right)$$

$$= \boxed{32\text{ J}}$$

(2) The path of a particle is described by $\vec{r}(t) = (\cos(4t), \sin(4t), -3t)$.

(a) Compute the velocity $\vec{v}(t)$ and acceleration $\vec{a}(t)$ of the particle.
(b) Compute the speed of the particle at time t.
(c) What angle is formed by the velocity vector and the unit vector $\hat{k} = (0, 0, 1)$ at time t?

a) $\vec{v}(t) = (-4\sin(4t), 4\cos(4t), -3)$

$\vec{a}(t) = (-16\cos(4t), -16\sin(4t), 0)$

b) speed $= |\vec{v}(t)| = \sqrt{16\sin^2(4t) + 16\cos^2(4t) + 9}$

$= \sqrt{16+9}$

$= \boxed{5}$

c) $|\vec{v}(t)| \cdot |\hat{k}| \cdot \cos\theta = \vec{v}(t) \cdot \hat{k}$

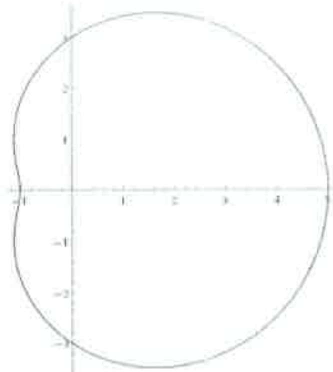
$= 0 + 0 + (-3) = -3.$

$|\vec{v}(t)| = 5$ and $|\hat{k}| = 1$

so $\cos\theta = -3/5$

$\boxed{\theta = \cos^{-1}\left(-\frac{3}{5}\right)}$

(3) The polar equation $r = 3 + 2\cos\vartheta$ has the graph shown.



(a) Find an equation in rectangular coordinates (x and y) for this curve.

(b) Compute the area enclosed by this curve.

a) $r^2 = 3r + 2\underbrace{r\cos\vartheta}_{=x}$

$$\boxed{x^2+y^2 = 3\sqrt{x^2+y^2} + 2x}$$

b) $\int_0^{2\pi} \frac{1}{2}(3+2\cos\vartheta)^2 \, d\vartheta$

$= \int_0^{2\pi} \frac{1}{2}(9 + 12\cos\vartheta + 4\cos^2\vartheta)\, d\vartheta$

$= \int_0^{2\pi} \frac{1}{2}(9 + 12\cos\vartheta + (2 + 2\cos 2\vartheta))\, d\vartheta$

$= \int_0^{2\pi} \frac{1}{2}(11 + 12\cos\vartheta + 2\cos 2\vartheta)\, d\vartheta$

$= \left[\frac{11}{2}\vartheta + 6\sin\vartheta + \frac{1}{2}\sin 2\vartheta\right]_0^{2\pi}$

$= \boxed{11\pi}$

(4) **CHOOSE ONE** of the following two problems. Indicate your choice by checking the box. Either choice will be worth the same number of points. **Do not solve both problems.** (Cross out any scratch work for the problem you decide not to submit)

☐ Find a nonzero real function $f(x)$ satisfying the following differential equation.

$$f''(x) + 2f'(x) + 10f(x) = 0$$

☐ Evaluate $\displaystyle\int_0^1 \frac{x^4}{(2-x^2)^{5/2}}dx$.

First option

Char. eqn. $\lambda^2 + 2\cdot\lambda + 10 = 0$

$$\Rightarrow \lambda = \frac{-2 \pm \sqrt{4 - 4\cdot 10}}{2} = -1 \pm \tfrac{1}{2}\sqrt{-36}$$

$$\lambda = -1 \pm i\cdot\sqrt{9}$$
$$= -1 \pm 3i$$

So one complex sol'n is $e^{(-1+3i)x}$.

In rect. form this is $e^{-x}\cos(3x) + i\cdot e^{-x}\sin(3x)$.

So one real solution is $\boxed{f(x) = e^{-x}\cos(3x)}$

(another is $e^{-x}\sin(3x)$)

Second option

$x = \sqrt{2}\sin\vartheta$ $\qquad \sqrt{2-x^2} = \sqrt{2}\cos\vartheta$

$dx = \sqrt{2}\cos\vartheta$ $\qquad \vartheta = \sin^{-1}(x/\sqrt{2})$

$$\int_0^1 \frac{x^4}{(2-x^2)^{5/2}} = \int_0^{\pi/4} \frac{(\sqrt{2})^4\sin^4\vartheta}{(\sqrt{2})^5\cos^5\vartheta}\cdot\sqrt{2}\cos\vartheta\, d\vartheta$$

$$= \int_0^{\pi/4} \frac{\sin^4\vartheta}{\cos^4\vartheta}\,d\vartheta$$

$$= \int_0^{\pi/4} \tan^4\vartheta\, d\vartheta$$

$$= \int_0^{\pi/4} (\sec^2\vartheta - 1)\cdot\tan^2\vartheta\, d\vartheta$$

$$= \int_0^{\pi/4} [\sec^2\vartheta\tan^2\vartheta - \tan^2\vartheta]\, d\vartheta$$

$$= \int_0^{\pi/4} [\sec^2\vartheta\tan^2\vartheta - \sec^2\vartheta + 1]\, d\vartheta$$

$$= \left[\tfrac{1}{3}\tan^3\vartheta - \tan\vartheta + \vartheta\right]_0^{\pi/4}$$

$$= \left(\tfrac{1}{3} - 1 + \tfrac{\pi}{4}\right) - (0 - 0 + 0) = \boxed{\frac{\pi}{4} - \frac{2}{3}}$$

(additional space for work)