MATH 19
MIDTERM 2 PRACTICE
14 NOVEMBER 2014

Name : Solutions

Show all of your reasoning. You may use the back of each page for additional space or scratch work. You do not need to simplify your answers unless specifically instructed to do so.

There are five problems. Each problem is worth 10 points.

You may use one page of notes (front and back). You do not need to submit it with the exam.

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(1) Solve the following initial value problem.

$$\begin{aligned} f'(x) &= f(x)\sec^2 x \\ f(0) &= 1 \end{aligned}$$

separate: $$\frac{f'(x)}{f(x)} = \sec^2 x$$

integrate: $$\int \frac{f'(x)}{f(x)}\,dx = \int \sec^2 x\,dx$$

$$\int \frac{df}{f} = \tan x + C$$

$$\ln|f(x)| = \tan x + C$$

solve: $$f(x) = \pm e^{\tan x + C}$$

use init. cond.:

$$1 = f(0) = \pm e^{\tan 0 + C} = \pm e^{C}$$

$\Rightarrow$ the "$\pm$" is a "$+$" and $C=0$.

$$\Rightarrow \quad \boxed{f(x) = e^{\tan x}}$$

(2) Determine whether or not each series converges. Be specific about which tests or facts you are using.

(a) $\sum_{m=1}^{\infty} (-1)^m \cdot \frac{m^7}{e^m}$

Ratio test:

$$\lim_{m\to\infty} \left| \frac{(-1)^{m+1}(m+1)^7/e^{m+1}}{(-1)^m m^7/e^m} \right| = \lim_{m\to\infty} \left(\frac{m+1}{m}\right)^7 \cdot \frac{1}{e}$$

$$= \left[\lim_{m\to\infty}\left(\frac{m+1}{m}\right)\right]^7 \cdot \frac{1}{e} = \left[\lim_{m\to\infty}\left(1+\frac{1}{m}\right)\right]^7 \cdot \frac{1}{e}$$

$$= 1/e.$$

Since $L<1$, the series $\boxed{\text{converges.}}$

(b) $\sum_{k=10}^{\infty} \frac{1}{k\sqrt{\ln k}}$

Integral test

$$\int_{10}^{\infty} \frac{1}{x\sqrt{\ln x}}\,dx \qquad u = \ln x,\quad du = \frac{1}{x}\,dx$$

$$= \int_{\ln 10}^{\infty} \frac{1}{\sqrt{u}}\,du = \Big[2\sqrt{u}\Big]_{\ln 10}^{\infty}$$

$$= \infty - 2\sqrt{\ln 10} = \infty.$$

Since the integral diverges, the sum also $\boxed{\text{diverges.}}$

(3) Solve the following initial value problem.

$$\begin{aligned} f''(t) + 6f'(t) + 9f(t) &= 27 \\ f(0) &= 4 \\ f'(0) &= -1 \end{aligned}$$

$f'' + 6f' + 9(f-3) = 0$

let $u = f - 3$.

Then $u'' + 6u' + 9u = 0$ (homogeneous)

char. poly $\lambda^2 + 6\lambda + 9 = 0$

$(\lambda+3)^2 = 0$

sol'n $\lambda = -3$ (repeated root)

$\Rightarrow$ gen'l sol'n of diffEq is

$u(t) = C \cdot e^{-3t} + D \cdot t \cdot e^{-3t}$

$f(t) = 3 + Ce^{-3t} + Dte^{-3t}$

$f'(t) = -3Ce^{-3t} + De^{-3t} - 3Dte^{-3t}$

init. conds:

$4 = f(0) = 3 + C$

$-1 = f'(0) = -3C + D$

$C = 1$

$D = 3C - 1 = 2$

$\Rightarrow \boxed{f(t) = 3 + e^{-3t} + 2te^{-3t}}$

(4) Evaluate the sum of each of the following series. You do not need to show that it converges.

(a) $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$

geo. series with first term 2/3 and ratio 2/3.

$$\text{sum} = \frac{2/3}{1-2/3} = \frac{2/3}{1/3} = \boxed{2}$$

(b) $\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{2}{3}\right)^n$

replace 2/3 by x:

$$\sum_{n=1}^{\infty} \frac{1}{n} x^n = \int_0^x \left(\sum_{n=1}^{\infty} t^{n-1}\right) dt$$

$$= \int_0^x \left(\frac{1}{1-t}\right) dt \qquad \text{(geo. series w/ 1}^{\text{st}}\text{ term 1 \& ratio } t\text{)}$$

$$= \Big[-\ln|1-t|\Big]_0^x$$

$$= -\ln|1-x|$$

setting x = 2/3 again:

$$\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{2}{3}\right)^n = -\ln\left(1-\frac{2}{3}\right) = -\ln\left(\frac{1}{3}\right)$$

$$= \boxed{\ln 3}$$

(5) **CHOOSE ONE** of the following two problems. Indicate your choice by checking the box. Either choice will be worth the same number of points. **Do not solve both problems.** (Cross out any scratch work for the problem you decide not to submit)

- [x] Find the third order Taylor approximation of $\tan x$ around $x = 0$.
- [ ] Determine whether or not the series $\sum_{n=1}^{\infty} \frac{n^n}{3^n \cdot n!}$ converges.

(other option on following page)

$f(x) = \tan x$

$f'(x) = \sec^2 x$

$f''(x) = 2\sec x \cdot \sec x \tan x$
$= 2\sec^2 x \tan x$

$f'''(x) = 2\cdot(2\sec x \cdot \sec x \tan x)\cdot \tan x + 2\sec^2 x \cdot \sec^2 x$
$= 4\sec^2 x \tan^2 x + 2\sec^4 x$

So
$f(0) = 0$
$f'(0) = 1$
$f''(0) = 0$
$f'''(0) = 2$

$P_3(x) = 0 + 1\cdot x + \frac{0}{2}\cdot x^2 + \frac{2}{3!}x^3$

$$\boxed{P_3(x) = x + \frac{1}{3}x^3}$$

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(other option on previous page)

Ratio test

$$L = \lim_{n\to\infty} \frac{(n+1)^{n+1}/(3^{n+1}\cdot(n+1)!)}{n^n/(3^n \cdot n!)}$$

$$= \lim_{n\to\infty} \frac{(n+1)^{n+1}}{n^n}\cdot\frac{3^n}{3^{n+1}}\cdot\frac{n!}{(n+1)!} = \lim_{n\to\infty}\left[\frac{1}{3}\cdot\frac{(n+1)^{n+1}}{n^n}\cdot\frac{1}{n+1}\right]$$

$$= \frac{1}{3}\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n$$

$$= \frac{1}{3} e^{\lim_{n\to\infty}\left[n\cdot \ln\left(\frac{n+1}{n}\right)\right]}$$

and $\lim_{n\to\infty}\left[n\cdot\ln\left(\frac{n+1}{n}\right)\right] = \lim_{n\to\infty}\frac{\ln\left(\frac{n+1}{n}\right)}{1/n}$ (indet. form $\frac{0}{0}$)

$$= \lim_{n\to\infty}\frac{\frac{n}{n+1}\left(\frac{1\cdot n-(n+1)\cdot 1}{n^2}\right)}{-1/n^2} \quad (\text{l'hôpital})$$

$$= \lim_{n\to\infty}\frac{\frac{n}{n+1}\cdot\frac{-1}{n^2}}{-1/n^2} = \lim_{n\to\infty}\frac{n}{n+1} = 1$$

thus $L = \frac{1}{3}\cdot e^1 = e/3$. So ($e\approx 2.7$), $L<1$

and this series $\boxed{\text{converges.}}$

(additional space for work)