MATH 19
MIDTERM 2
14 NOVEMBER 2014

Name : Solutions

Show all of your reasoning. You may use the back of each page for additional space or scratch work. You do not need to simplify your answers unless specifically instructed to do so.

There are five problems. Each problem is worth 10 points.

You may use one page of notes (front and back). You do not need to submit it with the exam.

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(1) Solve the following initial value problem.

$$\begin{aligned} \frac{dy}{dt} &= (1+y^2)\cos t \\ y(0) &= \sqrt{3} \end{aligned}$$

separate: $\frac{1}{1+y^2}\,dy = \cos t\,dt$

integrate: $\int \frac{1}{1+y^2}\,dy = \int \cos t\,dt$

$$\tan^{-1} y = \sin t + C$$

solve: $y = \tan(\sin t + C)$ (gen'l sol'n to the diffEq)

using the init. conditions:

$$\sqrt{3} = \tan(\sin 0 + C)$$

$$= \tan C$$

$\Rightarrow \quad C = \pi/3.$

$$\boxed{y = \tan(\sin t + \pi/3)}$$

(2) Determine whether or not each series converges. Be specific about which tests or facts you are using.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{n^2}$

Ratio test

$$L = \lim_{n\to\infty} \frac{(n+1)!\,/\,(n+1)^2}{n!\,/\,n^2}$$

$$= \lim_{n\to\infty} \frac{(n+1)!}{n!}\cdot\left(\frac{n}{n+1}\right)^2$$

$$= \lim_{n\to\infty} (n+1)\cdot\left(\frac{n}{n+1}\right)^2 = \infty \quad \left(\text{since } n+1\to\infty \text{ and } \left(\frac{n}{n+1}\right)^2 \to 1\right)$$

since $L>1$, the series $\boxed{\text{diverges.}}$

(b) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt{n}+6}$

Alt. series test: the series alternates,

$$\frac{1}{\sqrt{n+1}+6} < \frac{1}{\sqrt{n}+6} \quad \text{since } \sqrt{n+1} > \sqrt{n},$$

and $\displaystyle\lim_{n\to\infty}\frac{1}{\sqrt{n}+6} = \frac{1}{\infty} = 0$

so the series $\boxed{\text{converges.}}$

(3) Evaluate each of the following sums.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{3^{n-1}}{4^{n-1}}$

Geometric series with first term $\frac{3^0}{4^0} = 1$ and common ratio $3/4$

$$\Rightarrow \text{sum} = \frac{1}{1-3/4} = \frac{1}{1/4} = \boxed{4}$$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n \cdot 3^{n-1}}{4^{n-1}}$

Replacing $\left(\frac{3}{4}\right)$ by a variable x:

$$\sum_{n=1}^{\infty} n \cdot x^{n-1} = \frac{d}{dx}\left(\sum_{n=1}^{\infty} x^n\right)$$

$$= \frac{d}{dx}\left(\frac{x}{1-x}\right) \quad \text{(geo. series w/ first term } x \text{ and common ratio } x\text{)}$$

$$= \frac{1\cdot(1-x) - x\cdot(-1)}{(1-x)^2}$$

$$= \frac{1}{(1-x)^2}$$

Hence setting $x = 3/4$ again:

$$\sum_{n=1}^{\infty} n \cdot \left(\frac{3}{4}\right)^{n-1} = \frac{1}{(1-3/4)^2} = \frac{1}{(1/4)^2}$$

$$= \boxed{16}$$

(4) Solve the following initial value problem.

$$\begin{aligned} y''(t) + 6y'(t) + 10y(t) &= 20 \\ y(0) &= 3 \\ y'(0) &= 0 \end{aligned}$$

$y'' + 6y' + 10(y-2) = 0.$

Let $u = y-2$. Then

$$u'' + 6u' + 10u = 0 \qquad \text{(homog.)}$$

which has char. eqn. $\lambda^2 + 6\lambda + 10 = 0$

solutions $\lambda = \dfrac{-6 \pm \sqrt{36-40}}{2} = \dfrac{-6 \pm \sqrt{-4}}{2} = \dfrac{-6 \pm 2i}{2}$

$= -3 \pm i$

$\Rightarrow$ complex sol'n $e^{(-3+i)t} = e^{-3t}(\cos t + i \sin t)$

$\Rightarrow$ gen'l (real) sol'n is

$$u(t) = C \cdot e^{-3t}\cos t + D \cdot e^{-3t}\sin t$$

$$\Rightarrow y(t) = 2 + C \cdot e^{-3t}\cos t + D \cdot e^{-3t}\sin t$$

$$\begin{aligned} y'(t) = &-3Ce^{-3t}\cos t - C \cdot e^{-3t}\sin t \\ &-3De^{-3t}\sin t + D \cdot e^{-3t}\cos t \end{aligned}$$

using the initial conditions:

$$\left.\begin{aligned} 3 &= 2 + C \\ 0 &= -3C + D \end{aligned}\right| \quad \begin{aligned} C &= 1 \\ D &= 3C = 3 \end{aligned}$$

hence $\boxed{y(t) = 2 + e^{-3t}\cos t + 3e^{-3t}\sin t}$

(5) **CHOOSE ONE** of the following two problems. Indicate your choice by checking the box. Either choice will be worth the same number of points. **Do not solve both problems.** (Cross out any scratch work for the problem you decide not to submit)

- ☑ Find the third order Taylor approximation of $e^x \cos x$ around $x = 0$.
- ☐ Evaluate $\displaystyle\int_0^{\infty} e^{-2x} \sin x \; dx$.

(other option on next page)

$$f(x) = e^x \cos x$$
$$f'(x) = e^x \cos x - e^x \sin x$$
$$f''(x) = (e^x \cos x - e^x \sin x) - (e^x \sin x + e^x \cos x)$$
$$= -2e^x \sin x$$
$$f'''(x) = -2e^x \sin x - 2e^x \cos x$$

Hence:

$$f(0) = 1$$
$$f'(0) = 1 - 0 = 1$$
$$f''(0) = 0$$
$$f'''(0) = -2 \cdot 0 - 2 \cdot 1 = -2$$

And

$$P_3(x) = f(0) + f'(0) \cdot x + f''(0) \cdot \frac{1}{2} x^2 + f'''(0) \cdot \frac{1}{3!} x^3$$
$$= 1 + x + 0 \cdot x^2 - \frac{2}{6} \cdot x^3$$
$$\boxed{P_3(x) = 1 + x - \frac{1}{3} x^3}$$

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- ☑ Evaluate $\displaystyle\int_0^\infty e^{-2x} \sin x \; dx$.

(other option on previous page)

$$\int_0^\infty e^{-2x} \sin x\,dx \qquad u = e^{-2x} \quad dv = \sin x\,dx$$
$$du = -2e^{-2x} \quad v = -\cos x$$

$$= \left[-e^{-2x}\cos x\right]_0^\infty - \int_0^\infty 2e^{-2x}\cos x\,dx \qquad u = 2e^{-2x} \quad dv = \cos x\,dx$$
$$du = -4e^{-2x} \quad v = \sin x$$

$$= \lim_{x\to\infty}\left(-e^{-2x}\cos x\right) + e^0 \cdot \cos 0 - \left[2e^{-2x}\sin x\right]_0^\infty + \int_0^\infty (-4e^{-2x})\sin x\,dx$$

$$= 0 + 1 - \lim_{x\to\infty}\left(2e^{-2x}\sin x\right) + 2\cdot e^0 \sin 0 - 4\int_0^\infty e^{-2x}\sin x\,dx$$

$$\int_0^\infty e^{-2x}\sin x\,dx = 1 - 4\int_0^\infty e^{-2x}\sin x\,dx$$

$$\Rightarrow \quad 5\int_0^\infty e^{-2x}\sin x\,dx = 1$$

$$\Rightarrow \quad \int_0^\infty e^{-2x}\sin x\,dx = \boxed{\frac{1}{5}}$$

(additional space for work)