MATH 19
MIDTERM 1 PRACTICE
6 OCTOBER 2014

Name : Solutions

Show all of your reasoning. You may use the back of each page for additional space or scratch work. You do not need to simplify your answers unless specifically instructed to do so. Each question is worth 10 points.

You may use one page of notes (front and back), which you must submit with the exam. No calculators or other aids are permitted.

1		3	
2		4	
		Total	

(1) Evaluate the following integral.

$$\int_0^{4\pi} e^{-3x}\sin(2x)\;dx$$

$u = e^{-3x}$ $\qquad dv = \sin(2x)dx$

$du = -3e^{-3x}dx$ $\qquad v = -\frac{1}{2}\cos(2x)$

$$= \left[-\tfrac{1}{2}e^{-3x}\cdot\cos(2x)\right]_0^{4\pi} - \int_0^{4\pi}\left(-\tfrac{1}{2}\cos 2x\right)\left(-3e^{-3x}\right)dx$$

$$= -\tfrac{1}{2}e^{-12\pi} + \tfrac{1}{2}e^{0} - \int_0^{4\pi}\tfrac{3}{2}\cdot e^{-3x}\cos 2x\,dx$$

$u = e^{-3x}$ $\qquad dv = \cos 2x\,dx$

$du = -3e^{-3x}$ $\qquad v = \frac{1}{2}\sin(2x)$

$$= \tfrac{1}{2} - \tfrac{1}{2}e^{-12\pi} - \underbrace{\cancel{\tfrac{3}{2}\left[\tfrac{1}{2}e^{-3x}\cdot\sin(2x)\right]_0^{4\pi}}}_{0} + \tfrac{3}{2}\int_0^{4\pi}\left(\tfrac{1}{2}\sin(2x)\right)\cdot\left(-3e^{-3x}\right)dx$$

$$= \tfrac{1}{2} - \tfrac{1}{2}e^{-12\pi} + \tfrac{3}{2}\cdot\left(-\tfrac{3}{2}\right)\cdot\int_0^{4\pi} e^{-3x}\sin(2x)\,dx$$

So

$$\int_0^{4\pi} e^{-3x}\sin(2x)dx = -\tfrac{9}{4}\cdot\int_0^{4\pi} e^{-3x}\sin(2x)dx + \tfrac{1}{2} - \tfrac{1}{2}e^{-12\pi}$$

$$\Rightarrow \underbrace{\left(1+\tfrac{9}{4}\right)}_{13/4}\int_0^{4\pi} e^{-3x}\sin(2x)\,dx = \tfrac{1}{2} - \tfrac{1}{2}e^{-12\pi}$$

$$\Rightarrow \int_0^{4\pi} e^{-3x}\sin(2x)\,dx = \tfrac{4}{13}\cdot\tfrac{1}{2}\cdot\left(1-e^{-12\pi}\right)$$

$$= \boxed{\tfrac{2}{13}\left(1-e^{-12\pi}\right).}$$

(2) A particle moves at constant speed from position $(0,2,0)$ at time $t=0$ to position $(5,7,-5)$ at time $t=10$.

(a) Write the position $\vec{r}(t)$ of the particle as a function of t.

(b) Find the distance between the particle and the point $(5,0,-2)$, as a function of t.

a) $$\text{velocity} = \frac{\text{displacement}}{\text{time}}$$

$$= \frac{(5,7,-5)-(0,2,0)}{10}$$

$$= \frac{(5,5,-5)}{10}$$

$$= \left(\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}\right).$$

So $$\vec{r}(t) = (0,2,0) + t\cdot\left(\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2}\right)$$

$$= \boxed{\left(\tfrac{1}{2}t, \tfrac{1}{2}t+2, -\tfrac{1}{2}t\right)}$$

b) $$|\vec{r}(t) - (5,0,-2)|$$

$$= \left|\left(\tfrac{1}{2}t-5, \tfrac{1}{2}t+2-0, -\tfrac{1}{2}t+2\right)\right|$$

$$= \boxed{\sqrt{\left(\tfrac{1}{2}t-5\right)^2 + \left(\tfrac{1}{2}t+2\right)^2 + \left(-\tfrac{1}{2}t+2\right)^2}}$$

(3) Consider the polar curve defined by $r = \sqrt{2 + \cos\theta}$.

(a) Convert this equation to rectangular coordinates.

(b) Write an integral to compute the arc length of the curve (from 0 to 2π). Do not attempt to evaluate this integral.

a) $r^2 = 2 + \cos\theta$

$r^3 = 2r + \underbrace{r\cdot\cos\theta}_{x}$

$r^3 - 2r = x$

$r\cdot(r^2-2) = x$ $\boxed{\sqrt{x^2+y^2}\cdot(x^2+y^2-2) = x}$

b) $\frac{dr}{d\theta} = \frac{1}{2\sqrt{2+\cos\theta}}\cdot(-\sin\theta) = -\frac{\sin\theta}{2\sqrt{2+\cos\theta}}$

arc-length $= \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

$= \boxed{\int_0^{2\pi} \sqrt{(2+\cos\theta) + \frac{\sin^2\theta}{4(2+\cos\theta)}}\, d\theta}$

(4) **CHOOSE ONE** of the following two problems. Indicate your choice by checking the box. Either choice will be worth the same number of points. **Do not solve both problems.**

☐ Find a nonzero real function $f(x)$ satisfying the following differential equation.

$$f^{(4)}(x) + 4f(x) = 0$$

Here $f^{(4)}(x)$ denotes the fourth derivative of $f(x)$.

☐ Evaluate

$$\int_0^2 \frac{x^4}{\sqrt{4-x^2}}\,dx$$

First option

char. equation: $\lambda^4 + 4 = 0$

$\lambda^4 = -4$

$= 4 \cdot e^{i\pi}$

one fourth root of $4 \cdot e^{i\pi}$ is $\sqrt{2} \cdot e^{i\pi/4}$

$= \sqrt{2} \cdot \left(\frac{1}{\sqrt{2}} + i \cdot \frac{1}{\sqrt{2}}\right)$

$= 1 + i$

so $e^{(1+i)x} = e^x \cdot \cos x + i e^x \sin x$ is a complex solution

$\Rightarrow$ $\boxed{e^x \cos x}$ is one solution.

(other options are $e^x \sin x$, $e^{-x} \cos x$, $e^{-x} \sin x$).

Second option

$x = 2\sin\vartheta$

$\sqrt{4-x^2} = 2\cos\vartheta$

$dx = 2\cos\vartheta\, d\vartheta$

$$\int_0^2 \frac{x^4}{\sqrt{4-x^2}}\,dx = \int_0^{\pi/2} \frac{2^4 \sin^4\vartheta}{2 \cdot \cos\vartheta} \cdot 2\cos\vartheta\, d\vartheta$$

$$= 16 \int_0^{\pi/2} \sin^4\vartheta\, d\vartheta$$

$$= 16 \int_0^{\pi/2} \left[\frac{1}{2}(1 - \cos(2\vartheta))\right]^2 d\vartheta$$

(additional space for work)

Second option, continued

$$= 16 \cdot \frac{1}{4} \int_0^{\pi/2} (1-\cos(2\vartheta))^2 \, d\vartheta$$

$$= 4 \int_0^{\pi/2} (1 - 2\cos(2\vartheta) + \cos^2(2\vartheta)) \, d\vartheta$$

$$= 4 \int_0^{\pi/2} \left(1 - 2\cos(2\vartheta) + \frac{1}{2}(1+\cos(4\vartheta))\right) d\vartheta$$

$$= 4 \int_0^{\pi/2} \left(\frac{3}{2} - 2\cos(2\vartheta) + \frac{1}{2}\cos(4\vartheta)\right) d\vartheta$$

$$= 4 \cdot \left[\frac{3}{2}\vartheta - \sin(2\vartheta) + \frac{1}{8}\sin(4\vartheta)\right]_0^{\pi/2}$$

$$= 4\left(\frac{3}{2} \cdot \frac{\pi}{2} - 0 + 0\right) = \boxed{3\pi}$$