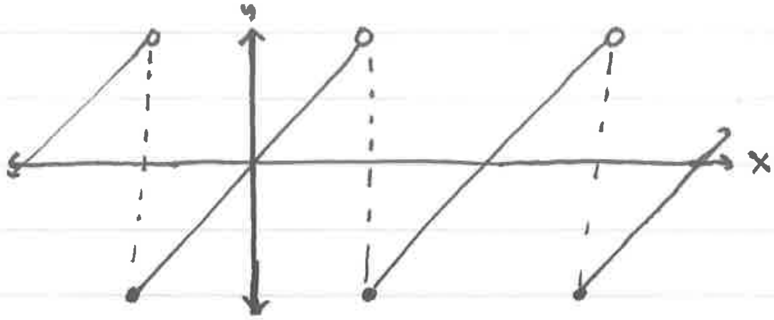
$\boxed{\text{P. Set 10 Solutions}}$

(1)



$$a_0 = \frac{1}{2}\int_{-1}^{1} x\,dx = \left[\frac{1}{4}x^2\right]_{-1}^{1} = \underline{\underline{0}}$$

For $n \geq 1$:

$$a_n = \int_{-1}^{1} x\cdot\cos(n\pi x)\,dx \qquad \begin{matrix} u = x \\ du = dx \end{matrix} \qquad \begin{matrix} dv = \cos(n\pi x)\,dx \\ v = \frac{1}{n\pi}\sin(n\pi x) \end{matrix}$$

$$= \left[x\cdot\frac{1}{n\pi}\sin(n\pi x)\right]_{-1}^{1} - \int_{-1}^{1}\frac{1}{n\pi}\sin(n\pi x)\,dx$$

$$= 1\cdot\frac{1}{n\pi}\cdot\sin(n\pi) - (-1)\cdot\frac{1}{n\pi}\cdot\sin(-n\pi) + \left[\frac{1}{n^2\pi^2}\cos(n\pi x)\right]_{-1}^{1}$$

$$= 0 + \frac{1}{n^2\pi^2}\left(\cos(n\pi) - \cos(-n\pi)\right)$$

$$= \underline{\underline{0}}$$ (or note $x\cdot\cos(n\pi x)$ is ~~even~~ odd & deduce immediately that the integral is 0).

$$b_n = \int_{-1}^{1} x\cdot\sin(n\pi x)\,dx \qquad \begin{matrix} u = x \\ du = dx \end{matrix} \qquad \begin{matrix} dv = \sin(n\pi x)\,dx \\ v = -\frac{1}{n\pi}\cos(n\pi x) \end{matrix}$$

$$= \left[-\frac{1}{n\pi}x\cdot\cos(n\pi x)\right]_{-1}^{1} + \frac{1}{n\pi}\int_{-1}^{1}\cos(n\pi x)\,dx$$

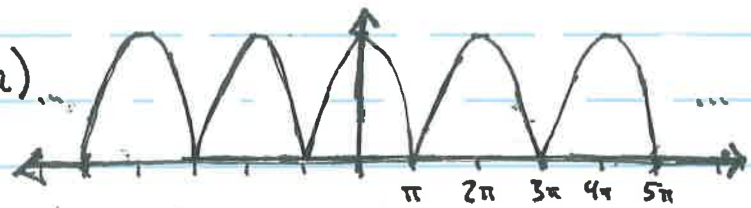
$$= -\frac{1}{n\pi}\cdot\left(1\cdot\cos(n\pi) - (-1)\cdot\cos(-n\pi)\right) + \frac{1}{n^2\pi^2}\left[\sin(n\pi x)\right]_{-1}^{1}$$

$$= -\frac{1}{n\pi}\cdot\left[(-1)^n + (-1)^n\right] + 0 = \underline{\underline{(-1)^{n-1}\cdot\frac{2}{n\pi}}}$$

Hence the Fourier series has only sines, and is given by:

$$\boxed{\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{2}{n\pi} \cdot \sin(n\pi x)}$$

(2) a)



the graph is a repeated pattern of parts of parabolas.

b) $$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} (\pi^2 - x^2)\,dx = \frac{1}{2\pi}\cdot\left[\pi^2 x - \frac{1}{3}x^3\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\cdot\left[\pi^3 - \frac{1}{3}\pi^3 + \pi^3 - \frac{1}{3}\pi^3\right] = \frac{1}{2\pi}\cdot\frac{4}{3}\pi^3$$

$$= \frac{2}{3}\pi^2$$

For $n \geq 1$:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi^2 - x^2)\cos(nx)\,dx \qquad \begin{matrix} u = \pi^2 - x^2 & dv = \cos(nx)dx \\ du = -2x & v = \frac{1}{n}\sin(nx)dx \end{matrix}$$

$$= \frac{1}{\pi}\left[(\pi^2 - x^2)\cdot\frac{1}{n}\sin(nx)\right]_{-\pi}^{\pi} + \frac{2}{n\pi}\int_{-\pi}^{\pi} x\cdot\sin(nx)\,dx$$

$$= \underbrace{\frac{1}{\pi}\cdot(\pi^2 - \pi^2)\cdot\frac{1}{n}\sin(n\pi) - \frac{1}{\pi}(\pi^2 - (-\pi)^2)\cdot\frac{1}{n}\sin(-n\pi)}_{0} + \frac{2}{n\pi}\int_{-\pi}^{\pi} x\cdot\sin(nx)\,dx$$

$$= \frac{2}{n\pi}\int_{-\pi}^{\pi} x\cdot\sin(nx)\,dx \qquad \begin{matrix} u = x & dv = \sin(nx)dx \\ du = dx & v = -\frac{1}{n}\cos(nx) \end{matrix}$$

$$= \frac{2}{n\pi}\cdot\left[-\frac{1}{n}x\cdot\cos(nx)\right]_{-\pi}^{\pi} + \frac{2}{n^2\pi}\int_{-\pi}^{\pi}\cos(nx)\,dx$$

$$= -\frac{2}{n^2\pi}\left[\pi\cdot\cos(n\pi) - (-\pi)\cdot\cos(-n\pi)\right] + \left[\frac{2}{n^3\pi}\sin(nx)\right]_{-\pi}^{\pi}$$

$$= -\frac{2}{n^2\pi}\left[\pi\cdot(-1)^n + \pi\cdot(-1)^n\right] + [0-0]$$

$$= (-1)^{n-1}\frac{4\pi}{n^2\pi} \qquad = \qquad \underline{\underline{(-1)^{n-1}\cdot\frac{4}{n^2}}}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(\pi^2-x^2)\sin(nx)\,dx \qquad \begin{matrix} u=\pi^2-x^2 & dv=\sin(nx)dx \\ du=-2x & v=-\frac{1}{n}\cos(nx)\end{matrix}$$

$$= \frac{1}{\pi}\left[(\pi^2-x^2)\cdot\left(-\frac{1}{n}\right)\cos(nx)\right]_{-\pi}^{\pi} - \frac{2}{n\pi}\int_{-\pi}^{\pi}x\cdot\cos(nx)\,dx$$

$$= \frac{1}{\pi}\left[\underbrace{(\pi^2-\pi^2)}_{0}\cdot\left(-\frac{1}{n}\right)\cdot\cos(n\pi) - \underbrace{(\pi^2-(-\pi)^2)}_{0}\left(-\frac{1}{n}\right)\cos(-n\pi)\right] - \frac{2}{n\pi}\int_{-\pi}^{\pi}x\cos(nx)dx$$

$$= -\frac{2}{n\pi}\int_{-\pi}^{\pi}x\cdot\cos(nx)\,dx \qquad \begin{matrix} u=x & dv=\cos(nx)dx \\ du=dx & v=\frac{1}{n}\sin(nx)\end{matrix}$$

$$= -\frac{2}{n\pi}\left[\frac{1}{n}x\sin(nx)\right]_{-\pi}^{\pi} + \frac{2}{n^2\pi}\int_{-\pi}^{\pi}\sin(nx)\,dx$$

$$= -\frac{2}{n\pi}\cdot\frac{1}{n}\cdot\pi\cdot\sin(n\pi) + \frac{2}{n\pi}(-\pi)\cdot\sin(-n\pi) + \frac{2}{n^2\pi}\cdot\left[-\frac{1}{n}\cos(nx)\right]_{-\pi}^{\pi}$$

$$= 0 + \frac{2}{n^2\pi}\cdot\left(-\frac{1}{n}\right)\cdot\cos(n\pi) - \frac{2}{n^2\pi}\cdot\left(-\frac{1}{n}\right)\cos(-n\pi)$$

$$= \frac{2}{n^2\pi}\cdot\left(-\frac{1}{n}\right)\cdot\left((-1)^n - (-1)^n\right) = \underline{\underline{0}}$$

(or just notice that $(\pi^2-x^2)\sin(nx)$ is an odd function)

So the Fourier series is

$$\boxed{\frac{2}{3}\pi^2 + \sum_{n=1}^{\infty}(-1)^{n-1}\cdot\frac{4}{n^2}\cdot\cos(nx)}$$

c) $f(0) = \pi^2$, so

$$\pi^2 = \frac{2}{3}\pi^2 + 4 \cdot \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{n^2} \cos(0)$$

$$= \frac{2}{3}\pi^2 + 4 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$$

$$\Rightarrow \quad \frac{1}{3}\pi^2 = 4 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}.$$

$$\Rightarrow \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \boxed{\frac{1}{12}\pi^2}$$

3 A formula for $f(x)$ is

$$f(x) = \begin{cases} 0 & -\pi \leq x \leq 0 \\ x & 0 \leq x < \pi \end{cases}, \quad \text{and}$$

$$f(x + 2\pi) = f(x).$$

Therefore :

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi} \int_0^{\pi} x\,dx$$

$$= \frac{1}{2\pi} \cdot \left[\frac{1}{2}x^2\right]_0^{\pi} = \frac{1}{2\pi} \cdot \frac{1}{2}\pi^2 = \underline{\underline{\frac{1}{4}\pi}}$$

for $n \geq 1$:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi} \int_0^{\pi} x \cdot \cos(nx)\,dx$$

$u = x \quad dv = \cos(nx)dx$

$du = dx \quad v = \frac{1}{n}\sin(nx)$

$$= \frac{1}{\pi} \cdot \left[\frac{1}{n} x \sin(nx)\right]_0^{\pi} - \frac{1}{n\pi} \int_0^{\pi} \sin(nx)\,dx$$

$$= \frac{1}{\pi}\cdot\left[\frac{1}{n}\pi\sin(n\pi) - \frac{1}{n}\cdot 0\cdot\sin 0\right] - \frac{1}{n\pi}\cdot\left[-\frac{1}{n}\cos(nx)\right]_0^{\pi}$$

$$= 0 - \frac{1}{n\pi}\left[-\frac{1}{n}(-1)^n + \frac{1}{n}\right]$$

$$= \underline{\underline{\frac{1}{n^2\pi}\left((-1)^n - 1\right)}} = \begin{cases} 0 & n \text{ even} \\ -2/n^2\pi & n \text{ odd} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\sin(nx)\,dx = \frac{1}{\pi}\int_0^{\pi} x\cdot\sin(nx)\,dx \qquad \begin{matrix} u = x & dv = \sin(nx)dx \\ du = dx & v = -\frac{1}{n}\cos(nx) \end{matrix}$$

$$= \frac{1}{\pi}\left[-\frac{1}{n}x\cos(nx)\right]_0^{\pi} + \frac{1}{n\pi}\int_0^{\pi}\cos(nx)\,dx$$

$$= \frac{1}{\pi}\left[-\frac{1}{n}\pi\cdot(-1)^n + \frac{1}{n}\cdot 0\cdot\cos(0)\right] + \left[\frac{1}{n^2\pi}\sin(nx)\right]_0^{\pi}$$

$$= \frac{(-1)^{n-1}}{n} + 0 + \frac{1}{n^2\pi}(0-0) = \underline{\underline{\frac{(-1)^{n-1}}{n}}}$$

So the Fourier series is

$$\boxed{\frac{1}{4}\pi + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2\pi}\cos(nx) + \frac{(-1)^{n-1}}{n}\sin(nx)\right]}$$

or alternatively

$$\boxed{\frac{1}{4}\pi - \sum_{n=0}^{\infty}\frac{2}{(2n+1)^2\pi}\cos\left((2n+1)x\right) + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sin(nx)}$$

4 a) $5 + 2\sin x + 3\cos(2x) = 5 + i e^{-ix} - i e^{ix} + \frac{3}{2}e^{-2ix} + \frac{3}{2}e^{2ix}$

$$= \boxed{\frac{3}{2}e^{-2ix} + i\cdot e^{-ix} + 5 - i\cdot e^{ix} + \frac{3}{2}e^{2ix}}$$

b) $1 - 4\cos x + 3\sin x = 1 - 2e^{-ix} - 2e^{ix} + \frac{3}{2}ie^{-ix} - \frac{3}{2}ie^{ix}$

$$= \boxed{(-2+\tfrac{3}{2}i)e^{-ix} + 1 + (-2-\tfrac{3}{2}i)e^{ix}}$$

c) $\frac{1}{2}\sin x + \frac{1}{4}\sin(2x) + \frac{1}{8}\sin(3x)$

$$= \frac{i}{4}e^{-ix} - \frac{i}{4}e^{-ix} + \frac{i}{8}e^{-2ix} - \frac{i}{8}e^{2ix} + \frac{i}{16}e^{-3ix} - \frac{i}{16}e^{3ix}$$

$$= \boxed{\frac{i}{16}e^{-3xi} + \frac{i}{8}e^{-2ix} + \frac{i}{4}e^{-ix} - \frac{i}{4}e^{ix} - \frac{i}{8}e^{2ix} - \frac{i}{16}e^{3ix}}$$

d) $6\cos x + 2\sin x + 5\sin(2x) + 3\cos(3x)$

$$= 3e^{-ix} + 3e^{ix} + ie^{-ix} - ie^{ix} + \frac{5}{2}ie^{-2ix} - \frac{5}{2}ie^{2ix} + \frac{3}{2}e^{-3ix} + \frac{3}{2}e^{3ix}$$

$$= \boxed{\frac{3}{2}e^{-3ix} + \frac{5}{2}ie^{-2ix} + (3+i)e^{-ix} + (3-i)e^{ix} - \frac{5}{2}ie^{2ix} + \frac{3}{2}e^{3ix}}$$

5 a) $5e^{-ix} + 5e^{ix} = 10\cdot\left[\frac{1}{2}e^{-ix} + \frac{1}{2}e^{ix}\right]$

$$= \boxed{10\cos x}$$

b) $(1+i)e^{-2ix} + (1-i)e^{2ix} = \left[e^{-2ix} + e^{2ix}\right] + \left[i\cdot e^{-2ix} - i\cdot e^{2ix}\right]$

$$= \boxed{2\cos(2x) + 2\sin(2x)}$$

c) $\frac{1}{1+2i}e^{-2ix} + \frac{1}{1+i}e^{-ix} + \frac{1}{1-i}e^{ix} + \frac{1}{1-2i}e^{2ix}$

$$= \frac{1}{1+2i}(\cos 2x - i\sin 2x) + \frac{1}{1+i}(\cos x - i\sin x)$$
$$+ \frac{1}{1-i}(\cos x + i\sin x) + \frac{1}{1-2i}(\cos 2x + i\sin 2x)$$

$$= \left[\frac{1}{1+i} + \frac{1}{1-i}\right]\cos x + \left[-\frac{i}{1+i} + \frac{i}{1-i}\right]\sin x$$

$$+ \left[\frac{1}{1+2i} + \frac{1}{1-2i}\right]\cos(2x) + \left[-\frac{i}{1+2i} + \frac{i}{1-2i}\right]\sin(2x)$$

$$= \frac{1-i+1+i}{(1+i)(1-i)}\cos x + \frac{-i(1-i)+i(1+i)}{(1+i)(1-i)}\sin x$$

$$+ \frac{1-2i+1+2i}{(1+2i)(1-2i)}\cos(2x) + \frac{-i(1-2i)+i(1+2i)}{(1+2i)(1-2i)}\sin(2x)$$

$$= \frac{2}{2}\cos x + \frac{-i-1+i-1}{2}\sin x + \frac{2}{5}\cos(2x) + \frac{-i-2+i-2}{5}\sin(2x)$$

$$= \boxed{\cos x - \sin x + \frac{2}{5}\cos(2x) - \frac{4}{5}\sin(2x)}$$

d) $ie^{-3ix} - e^{-2ix} - ie^{-ix} + 1 + ie^{ix} - e^{2ix} - ie^{3ix}$

$$= \left[ie^{-3ix} - ie^{3ix}\right] - \left[e^{-2ix} + e^{2ix}\right] - \left[ie^{-ix} - ie^{ix}\right] + 1$$

$$= 2\sin(3x) - 2\cos(2x) - 2\sin(x) + 1$$

$$= \boxed{1 - 2\sin x - 2\cos(2x) + 2\sin(3x)}$$

6 a) $Q'' + 2Q' + 5Q = V$

$\Rightarrow$ for all n, $\left((in)^2 + 2(in) + 5\right) C_n(Q) = C_n(V)$

i.e. $\boxed{C_n(Q) = \frac{1}{(in)^2 + 2(in) + 5} C_n(V)}$

b) $V(t) = 2\cos t + 2\cos 2t + 2\cos 3t$

$$= e^{-it} + e^{it} + e^{-2it} + e^{2it} + e^{-3it} + e^{3it}$$

$$= e^{-3it} + e^{-2it} + e^{-it} + e^{it} + e^{2it} + e^{3it}$$

hence $C_{-3} = C_{-2} = C_{-1} = C_1 = C_2 = C_3 = 1$, and all other c_n are 0. (for the function $V(t)$).

Therefore:

$$C_{-3}(Q) = \frac{1}{(-3i)^2 + 2i\cdot(-3) + 5} \cdot 1 = \frac{1}{-9-6i+5} = \frac{1}{-4-6i}$$

$$= \frac{-4+6i}{(-4-6i)(-4+6i)} = \frac{-4+6i}{16+36} = \frac{-4+6i}{52} = \underline{\underline{\frac{-2+3i}{26}}}$$

$$C_{-2}(Q) = \frac{1}{(-2i)^2 + 2i(-2) + 5} \cdot 1 = \frac{1}{-4-4i+5} = \frac{1}{1-4i}$$

$$= \frac{1+4i}{(1-4i)(1+4i)} = \frac{1+4i}{1+4^2} = \underline{\underline{\frac{1+4i}{17}}}$$

$$C_{-1}(Q) = \frac{1}{(-i)^2 + 2i(-1) + 5} \cdot 1 = \frac{1}{-1-2i+5} = \frac{1}{4-2i}$$

$$= \frac{4+2i}{(4-2i)(4+2i)} = \frac{4+2i}{16+4} = \frac{4+2i}{20} = \underline{\underline{\frac{2+i}{10}}}$$

$$C_1(Q) = \frac{1}{i^2 + 2i + 5} \cdot 1 = \frac{1}{4+2i} = \frac{4-2i}{(4+2i)(4-2i)}$$

$$= \frac{4-2i}{16+4} = \frac{4-2i}{20} = \underline{\underline{\frac{2-i}{10}}}$$

$$C_2(Q) = \frac{1}{(2i)^2 + 2i\cdot 2 + 5} = \frac{1}{-4+4i+5} = \frac{1}{1+4i} = \frac{1-4i}{(1+4i)(1-4i)}$$

$$= \frac{1-4i}{1+16} = \underline{\underline{\frac{1-4i}{17}}}$$

$$C_3(Q) = \frac{1}{(3i)^2+2i\cdot 3+5} = \frac{1}{-9+6i+5} = \frac{1}{-4+6i}$$

$$= \frac{-4-6i}{(-4+6i)(-4-6i)} = \frac{-4-6i}{16+36} = \frac{-4-6i}{52} = \underline{\underline{\frac{-2-3i}{26}}}$$

c) From the previous part,

$$Q(t) = \tfrac{1}{26}(-2+3i)e^{-3it} + \tfrac{1}{17}(1+4i)e^{-2it} + \tfrac{1}{10}(2+i)e^{-it}$$

$$+ \tfrac{1}{10}(2-i)e^{it} + \tfrac{1}{17}(1-4i)e^{2it} + \tfrac{1}{26}(-2-3i)e^{3it}$$

$$= \tfrac{2}{10}\left(e^{-it}+e^{it}\right) + \tfrac{i}{10}\left(e^{-it}-e^{it}\right)$$

$$+ \tfrac{1}{17}\left(e^{-2it}+e^{2it}\right) + \tfrac{4}{17}i\left(e^{-2it}-e^{2it}\right)$$

$$\cancel{+} - \tfrac{2}{26}\left(e^{-3it}+e^{3it}\right) + \tfrac{3}{26}i\left(e^{-3it}-e^{3it}\right)$$

$$= \tfrac{4}{10}\cos t + \tfrac{2}{10}\sin t + \tfrac{2}{17}\cos 2t + \tfrac{8}{17}\sin 2t$$
$$- \tfrac{4}{26}\cos 3t + \tfrac{6}{26}\sin 3t$$

$$= \tfrac{2}{5}\cos t + \tfrac{1}{5}\sin t + \tfrac{2}{17}\cos 2t + \tfrac{8}{17}\sin 2t$$
$$- \tfrac{2}{13}\cos 3t + \tfrac{3}{13}\sin(3t).$$

So the real Fourier coefficients are:

$a_1 = 2/5$	$b_1 = 1/5$
$a_2 = 2/17$	$b_2 = 8/17$
$a_3 = -2/13$	$b_3 = 3/13$

& the rest are 0.

(7) We know that if $f(x) = \sum a_n x^n$
then $\sum n \cdot a_n \cdot x^n = x \cdot f'(x)$.

Now,

$$\sum_{n=1}^{\infty} x^n = \frac{x}{1-x} \quad \text{(geo. series)}$$

$$\Rightarrow \sum_{n=1}^{\infty} n \cdot x^n = x \cdot \frac{d}{dx}\left(\frac{x}{1-x}\right)$$

$$= x \cdot \frac{1}{(1-x)^2} = \frac{x}{(1-x)^2}$$

applying the fact again:

$$\Rightarrow \sum_{n=1}^{\infty} n^2 \cdot x^n = x \cdot \frac{d}{dx}\left(\frac{x}{(1-x)^2}\right)$$

$$= x\left[\frac{1}{(1-x)^2} + \frac{2x}{(1-x)^3}\right] \quad \text{(product rule)}$$

$$= \frac{x}{(1-x)^2} + \frac{2x^2}{(1-x)^3}$$

$$= \frac{x(1-x)}{(1-x)^3} + \frac{2x^2}{(1-x)^3}$$

$$= \frac{x - x^2 + 2x^2}{(1-x)^3} = \boxed{\frac{x+x^2}{(1-x)^3}}$$

(8) Using problem 7, with $x = 0.999$,

$$\sum_{n=1}^{\infty} n^2 \cdot p_n = \sum_{n=1}^{\infty} n^2\, 0.999^{n-1} \cdot 0.001$$

$$= \frac{0.001}{0.999} \cdot \sum_{n=1}^{\infty} n^2\, 0.999^n$$

$$= \frac{0.001}{0.999} \cdot \frac{0.999 + 0.999^2}{(1-0.999)^3}$$

$$= \frac{\cancel{0.001}}{\cancel{0.999}} \cdot \frac{\cancel{0.999}(1+0.999)}{0.001^{\cancel{3}\,2}}$$

$$= \frac{1.999}{0.001^2} = 1.999 \cdot (1{,}000{,}000)$$

$$= 1{,}999{,}000.$$

Therefore

$$\sigma^2 = \sqrt{\sum_{n=1}^{\infty} n^2 p_n - \mu^2}$$

$$= \sqrt{1{,}999{,}000 - 1000^2}$$

$$= \boxed{\sqrt{999{,}000}} \approx 999.5$$