P. Set 4 Solutions

① a) $(r,\theta) = (-2, \pi/27) \longrightarrow \boxed{(r,\theta) = (2, -\frac{26}{27}\pi)}$

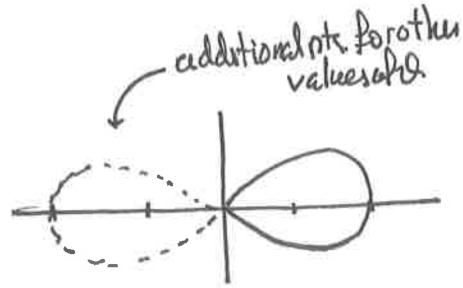
b) $(r,\theta) = (2, 3\pi) \longrightarrow \boxed{(r,\theta) = (2,\pi)}$

c) $(r,\theta) = (4, -\pi/9) \longrightarrow \boxed{(r,\theta) = (4, \frac{17}{9}\pi)}$

d) $(r,\theta) = (5, \frac{8}{7}\pi) \longrightarrow \boxed{(r,\theta) = (5, -\frac{6}{7}\pi)}$

(many other answers are possible for each part)

② $r = 2\sqrt{\cos 2\theta}$, $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$



a) $r^2 = 4\cos(2\theta)$

$= 4\cos^2\theta - 4\sin^2\theta$

$\Rightarrow r^4 = 4(r\cdot\cos\theta)^2 - 4\cdot(r\sin\theta)^2$

$\Rightarrow \boxed{(x^2+y^2)^2 = 4x^2 - 4y^2}$

Note. This equation gives more points than just those from $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$; these points appear for values with $\frac{3}{4}\pi \le \theta \le \frac{5}{4}\pi$. Other values of θ would not give real values of r.

b) Area $= \int_{-\pi/4}^{\pi/4} \frac{1}{2} r^2 \, d\theta = \int_{-\pi/4}^{\pi/4} \frac{1}{2}\cdot 4\cos 2\theta \, d\theta$

$= 2\int_{-\pi/4}^{\pi/4} \cos(2\theta)\, d\theta = \left[\sin(2\theta)\right]_{-\pi/4}^{\pi/4} = 2.$

c) $\frac{dr}{d\theta} = 2\cdot \frac{1}{2\sqrt{\cos 2\theta}} \cdot \frac{d}{d\theta}(\cos 2\theta) = \frac{1}{\sqrt{\cos 2\theta}}\cdot(-\sin 2\theta)\cdot 2 = -2\cdot\frac{\sin 2\theta}{\sqrt{\cos 2\theta}}$

So

$$r^2+\left(\frac{dr}{d\vartheta}\right)^2 = 4\cos 2\vartheta + 4\cdot\frac{\sin^2 2\vartheta}{\cos 2\vartheta} = 4\cdot\left(\frac{\cos^2 2\vartheta}{\cos 2\vartheta}+\frac{\sin^2 2\vartheta}{\cos 2\vartheta}\right)$$

$$= \frac{4}{\cos 2\vartheta} = 4\sec 2\vartheta$$

$$\Rightarrow \text{arc-length} = \int_{-\pi/4}^{\pi/4}\sqrt{4\sec 2\vartheta}\,d\vartheta = 2\int_{-\pi/4}^{\pi/4}\sqrt{\sec 2\vartheta}\,d\vartheta$$

$$= \int_{-\pi/2}^{\pi/2}\sqrt{\sec u}\,du \quad (\text{where } u=2\vartheta)$$

using a computer, this is $\approx \underline{5.244}$.

(3) $(x-A)^2+(y-B)^2 = A^2+B^2$

$$x^2-2xA+\cancel{A^2}+y^2-2yB+\cancel{B^2}=\cancel{A^2}+\cancel{B^2}$$

$$x^2+y^2-2xA-2yB=0$$

$$\Leftrightarrow \quad r^2-2r\cos\vartheta\cdot A-2r\sin\vartheta\cdot B=0$$

$$\Leftrightarrow \quad (r=0 \text{ or}) \quad r-2A\cos\vartheta-2B\sin\vartheta=0$$

$$\boxed{r=2A\cdot\cos\vartheta+2B\cdot\sin\vartheta}$$

This problem originally had a misprint, and read:

$$(x-A)^2+(x-B)^2=A^2+B^2$$

which is equivalent to

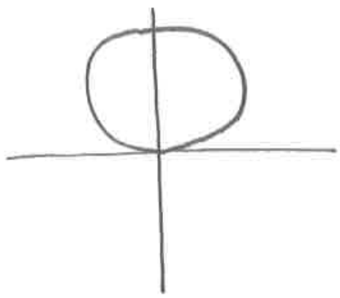
$$2x^2-2x(A+B)=0$$

$\Leftrightarrow$ $x=0$ or $x=A+B$, a pair of vertical lines.

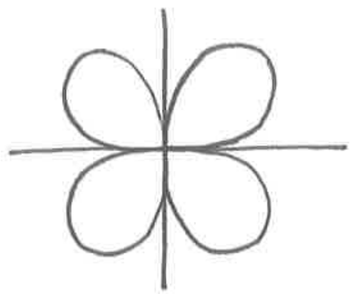
The second line has polar equation $r=(A+B)/\cos\vartheta$, while the first ($x=0$) has no equation of the form $r=F(\vartheta)$.

Due to the misprint, the graders will also accept $r=(A+B)/\cos\vartheta$.

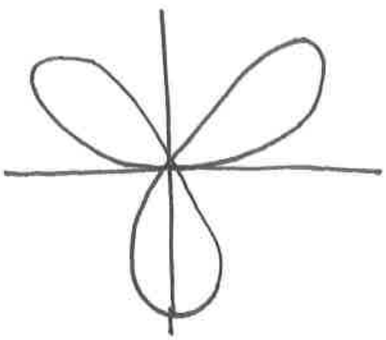
(4)



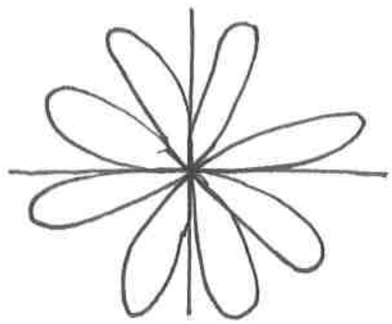
$n=1$



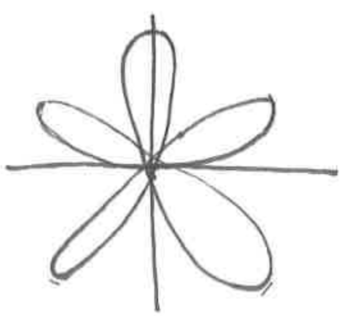
$n=2$



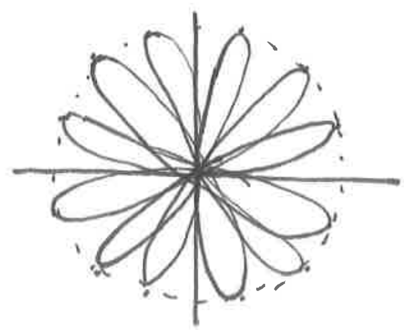
$n=3$



$n=4$



$n=5$



$n=6$

When <u>n is odd</u> there are <u>n petals</u>; when <u>n is even</u> there are <u>$2n$</u>.

[The reason for this is that the petals end at the $2n$ polar points $\left(1, \frac{\pi}{2n}\right), \left(-1, \frac{3\pi}{2n}\right), \left(1, \frac{5\pi}{2n}\right), \left(-1, \frac{7\pi}{2n}\right), \cdots, \left(1, \frac{(4n-3)\pi}{2n}\right), \left(-1, \frac{(4n-1)\pi}{2n}\right)$, but when n is odd these points contain repeats:

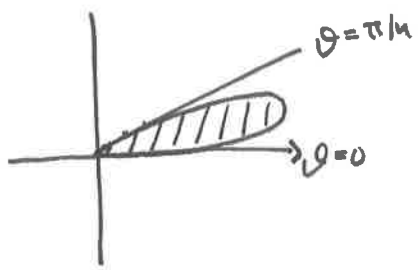
$\left(1, \frac{\pi}{2n}\right)$ is the same point as $\left(-1, \frac{(2n+1)\pi}{2n}\right)$

$\left(-1, \frac{3\pi}{2n}\right)$ " $\left(+1, \frac{(2n+3)\pi}{2n}\right)$

(etc.)

so each lobe is covered twice as θ ranges through $[0,2\pi)$.]

The area of one lobe is the area for $0 \le \theta \le \frac{\pi}{n}$, which is



$$\int_0^{\pi/n} \frac{1}{2} r^2 \, d\theta$$

$$= \frac{1}{2} \int_0^{\pi/n} \sin^2(n\theta)\, d\theta \qquad \begin{aligned} u &= n\theta \\ du &= n\,d\theta \end{aligned}$$

$$= \frac{1}{2n} \int_0^{\pi} \sin^2 u \, du$$

$$= \frac{1}{2n} \int_0^{\pi} \frac{1}{2}(1-\cos(2u))\, du$$

$$= \frac{1}{2n}\left[\frac{1}{2}u - \frac{1}{4}\sin 2u\right]_0^{\pi}$$

$$= \frac{1}{2n} \cdot \frac{1}{2}\pi = \boxed{\pi/4n}$$

Another method: $\frac{1}{2}\int_0^{2\pi} \sin^2(n\theta)\, d\theta = \frac{1}{2}\pi$, and there are $2n$ lobes covered from 0 to 2π (if n is odd there are n lobes, but each one is ~~cover~~ swept out twice), so one lobe has area $\frac{1}{2}\pi/2n = \pi/4n$.

(5)

A hyperbola has an equation of the form

$$r = \frac{1}{1 - e\cos\theta}$$

where $e > 1$.

The two "bad angles" occur when $1 - e\cos\theta = 0$, ie $\cos\theta = \frac{1}{e}$.

These angles give rays parallel to the asymptotes. Hence

$$\cos\left(\pm\frac{\pi}{4}\right) = \frac{1}{e}$$

$$\Rightarrow \quad \frac{1}{\sqrt{2}} = \frac{1}{e}$$

$$\Rightarrow \quad e = \sqrt{2}$$

So one equation for such a hyperbola is $\boxed{r = \frac{1}{1 - \sqrt{2}\cdot\cos\theta}}$.

Rectangular: $r - \sqrt{2}\cdot \underbrace{r\cdot\cos\theta}_{x} = 1 \Leftrightarrow r^2 = (1 + \sqrt{2}\,x)^2 \Leftrightarrow x^2 + y^2 = 2x^2 + 2\sqrt{2}\,x + 1$

$\Leftrightarrow \boxed{x^2 + 2\sqrt{2}\,x - y^2 + 1 = 0}$ or $\boxed{(x+\sqrt{2})^2 - y^2 = 1.}$

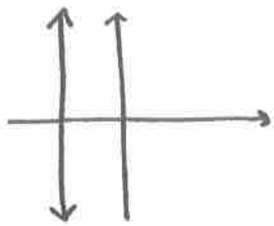
(6) $r = \frac{1}{A - \cos\theta} = \frac{1}{A} \cdot \frac{1}{1 - \frac{1}{A}\cdot\cos\theta}$, when $\underline{A \neq 0}$.

The case $\underline{A = 0}$ is special; in that case $r = \frac{1}{-\cos\theta}$ gives $x = -1$; the graph is just a vertical line.

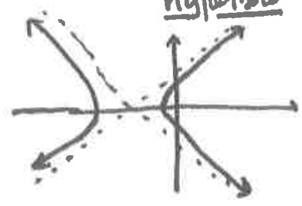
When $A \neq 0$, the curve is a conic section with eccentricity $e = 1/A$. The three cases we discussed in class were $e = 0$ (which doesn't occur here), $0 < e < 1$, $e = 1$, and $e > 1$. These correspond to $A > 1$, $A = 1$, and $0 < A < 1$. When A is negative, the curve turns out to be the same as with $-A$ (since $r(\theta + \pi) = \frac{1}{A - \cos(\theta + \pi)} = \frac{1}{A + \cos\theta} = -\frac{1}{-A - \cos\theta}$) so the shape depends only on $|A|$.

So the following cases are possible:

$A=0$: line

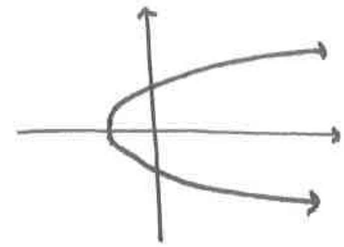


$0<A<1$ or $-1<A<0$: hyperbola

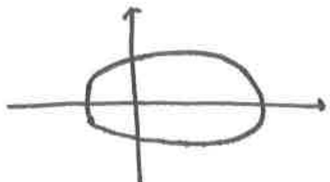


(since $\left|\frac{1}{A}\right|>1$)

$A=\pm 1$: parabola



$A>1$ or $A<-1$: ellipse



(since $\frac{1}{|A|}<1$)

7 a)

$z^8=1$ means $r^8\cdot e^{8i\theta}$ if $z=r\cdot e^{i\theta}$.

So $r=1$ and ~~8θ~~ $(1,8\theta)$ must be the same point as $(1,0)=(r,\theta)$.

The solutions are

$$z=1,\ e^{2\pi i/8},\ e^{4\pi i/8},\ \dots,\ e^{14\pi i/8}$$

ie. $\boxed{z=1,\ e^{i\pi/4},\ e^{i\pi/2},\ e^{i\cdot 3\pi/4},\ e^{i\pi},\ e^{i\cdot 5\pi/4},\ e^{i\cdot 3\pi/2},\ e^{i\cdot 7\pi/4}}$

or, in rectangular form, $\boxed{z=\pm 1 \text{ or } \pm i \text{ or } \pm\frac{1}{\sqrt{2}} \pm i\cdot\frac{1}{\sqrt{2}}}$.

b) $z^3=8i$ means $r^3\cdot e^{3i\theta}=8\cdot e^{i\pi/2}$.

So $r=2$ and θ is one of $\pi/6$, $\pi/6+\frac{2\pi}{3}=\frac{5}{6}\pi$, or $\pi/6+\frac{4\pi}{3}=\frac{3}{2}\pi$.

in polar form, $z=2e^{i\pi/6},\ 2e^{i\cdot 5\pi/6}$, or $2e^{i\cdot 3\pi/2}$

in rectangular form, $z=$ $\sqrt{3}+i$, $-\sqrt{3}+i$, $\cancel{\frac{\sqrt{3}}{2}+\frac{1}{2}i}$, $\cancel{-\frac{\sqrt{3}}{2}+\frac{1}{2}i}$, or $-2i$.

(8) Let $z=a+bi$, so $\bar{z}=a-bi$.

a) $z+\bar{z}=\underline{2a}$, so $\boxed{\text{any real number}}$ is possible.

b) $z-\bar{z}=2bi$, so $\boxed{\text{any imaginary number}}$ is possible.

c) $z\cdot\bar{z}=a^2+b^2$, so $\boxed{\text{any nonnegative real number}}$ is possible.

d) $$z/\bar{z}=\frac{a+bi}{a-bi}=\frac{(a+bi)^2}{a^2+b^2}=\frac{a^2+2abi-b^2}{a^2+b^2}=\frac{1}{a^2+b^2}\cdot(a^2-b^2+2abi).$$

or, in polar form, $\frac{re^{i\theta}}{r\cdot e^{-i\theta}}=\frac{z}{\bar{z}}=e^{2i\theta}$, which can be $\boxed{\text{any complex number of absolute value 1}}$ as θ varies.

(9) Begin with the second equation:

$$(1-i)z+(5+i)w=0$$

$$\Rightarrow \cancel{5}\ (1-i)z=-(5+i)w$$

$$\Rightarrow z=-\frac{5+i}{1-i}\cdot w=-\frac{(5+i)(1+i)}{(1-i)(1+i)}w$$

$$=-\frac{5+5i+i-1}{1^2+1^2}w=-\frac{4+6i}{2}w$$

$$=(-2-3i)w$$

substituting for z, in the first equation:

$$(1+i)(-2-3i)w+(\overset{2}{\cancel{5}}+i)w=4+3i$$

$$(-2-3i-2i+3)w+(2+i)w=4+3i$$

$$(1-5i)w+(2+i)w=4+3i$$

$$(3-4i)w=4+3i$$

$$w=\frac{4+3i}{3-4i}=\frac{(4+3i)(3+4i)}{3^2+4^2}=\frac{12+16i+9i-12}{25}$$

$$=i.$$

So $\quad w = i \ \ \& \ \ z = (-2-3i)w = (-2-3i)\cdot i = 3-2i$

ie. $\boxed{\begin{array}{l} z = 3-2i \\ w = i \end{array}}$

(10) Begin by finding f' & f'' in each case:

case	$f(x)$	$f'(x)$	$f''(x)$
A	e^{2x}	$2e^{2x}$	$4e^{2x}$
B	$x\cdot e^x$	$e^x + x\cdot e^x$	$2e^x + x\cdot e^x$
C	$e^{-x}\cos x$	$-e^{-x}\cos x - e^{-x}\sin x$	$2e^{-x}\sin x$
D	$e^x\cdot \sin x$	$e^x\sin x + e^x\cos x$	$2e^x\cos x$

Then compute each of the four expressions to see which is 0:

	I $f''-2f'+f$	II $f''+2f'+2f$	III $f''-f'-2f$	IV $f''-2f'+2f$
A	e^{2x}	$10e^{2x}$	0	$2e^{2x}$
B	0	$4e^x + 5xe^x$	$e^x - 2xe^x$	$x\cdot e^x$
C	$3e^{-x}\cos x + 4e^{-x}\sin x$	0	$-e^{-x}\cos x + 3e^{x}\sin x$	$4e^{-x}\cos x + 4e^{-x}\sin x$
D	$-e^x\sin x$	$4e^x\sin x + 4e^x\cos x$	$-3e^x\sin x + e^x\cos x$	0

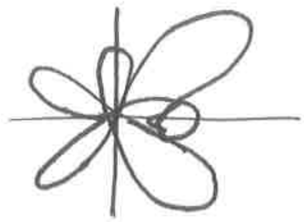
Therefore

A solves III
B solves I
C solves II
D solves IV

11

a) Changing the "12" appears to change the number of "layers" of the curve.

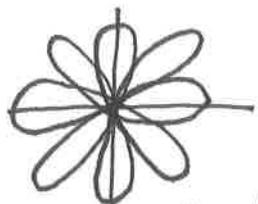
eg. making it a "1" produces



(rough sketch).

Changing to non-integer values can create many folds/layers criss-crossing each other.

b) Making the "2" larger tends to equalize the sizes of the various lobes. If "2" becomes "50", the butterfly looks like a flower:



(roughly).

~~Or~~ On the other hand, smaller values exaggerate the wings; the front lobes begin to bulge out.

c) There are many possible answers. For example:

- Replacing $e^{\cos\theta}$ by $0.5e^{\cos\theta}$ or ~~e~~ $0.1e^{\cos\theta}$ makes the lobes more balanced (more like a flower), ~~and~~ while replacing it with $2e^{\cos\theta}$ exaggerates the front lobes of the wings.
- Changing the exponent "5" shifts the layers ~~but~~ but preserves the basic shape of the butterfly.
- Changing the "4" (eg. to 2 or 6) dramatically changes the sizes & numbers of lobes (it no longer looks much like a butterfly).