P. Set 7 Solutions

(1)

a) $6+2+\frac{2}{3}+\cdots = \frac{6}{1-1/3} = \frac{6}{2/3} = \boxed{9}$

b) $\sum_{\ell=0}^{\infty} \frac{9}{10^\ell} = \frac{9}{1-1/10} = \boxed{10}$

c) $\sum_{\ell=1}^{\infty} \frac{9}{10^\ell} = \frac{9/10}{1-1/10} = \boxed{1}$

d) $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{9^k} = \frac{(-1)^0/9^1}{1-(-1/9)} = \frac{1/9}{1+1/9} = \boxed{\frac{1}{10}}$

(2)

a) $\sum_{n=0}^{\infty} \frac{(-1)^n}{x^n} = \frac{(-1)^0/x^0}{1+1/x} = \frac{1}{1+1/x} = \boxed{\frac{x}{x+1}}$ (converges when $|\frac{1}{x}|<1$)

b) $\sum_{\ell=0}^{\infty} (-1)^\ell x^{\ell+1} = \frac{(-1)^0 \cdot x^{0+1}}{1-(-x)} = \boxed{\frac{x}{x+1}}$ (conv. when $|x|<1$)

<u>Note</u>. (a) & (b) give two series expansions of the same function, but they converge for different values of x. This is often useful in practice: different series can be chosen according to which one will converge.

c) $\sum_{k=1}^{\infty} x^{3k} = \boxed{\frac{x^3}{1-x^3}}$ (first term is x^3, common ratio is x^3).

d) $\sum_{k=1}^{\infty} (-1)^{k-1} \cdot x^{3k} = \frac{(-1)^0 \cdot x^3}{1+x^3} = \boxed{\frac{x^3}{1+x^3}}$

(both (c) & (d) converge when $|x|<1$)

③ Because $\frac{d}{dx}(x^n) = n \cdot x^{n-1}$

it follows that

$$\sum_{n=1}^{\infty} n \cdot x^{n-1} = \frac{d}{dx}\left(\sum_{n=1}^{\infty} x^n\right)$$

$$= \frac{d}{dx}\left(\frac{x}{1-x}\right)$$

$$= \frac{(1-x) - x\cdot(-1)}{(1-x)^2}$$

$$= \boxed{\frac{1}{(1-x)^2}}$$

(where it converges)

④ From the previous problem:

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n \cdot x^{n-1}$$

Therefore,

multiplying by x: $\frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} x \cdot n\, x^{n-1}$

$$= \sum_{n=1}^{\infty} n \cdot x^n$$

replacing x by x^2 on both sides: $\boxed{\frac{x^2}{(1-x^2)^2}} = \sum_{n=1}^{\infty} n \cdot x^{2n}$

(where it converges)

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We saw before that

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n \cdot x^{n-1} \qquad \text{(when it converges)}$$

$$\Rightarrow \quad \frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} n \cdot x^n$$

Therefore, assuming this converges for $x = 2/3$,

$$\sum_{n=1}^{\infty} n \cdot \left(\tfrac{2}{3}\right)^n = \frac{2/3}{(1-2/3)^2}$$

$$= \frac{2/3}{1/9}$$

$$= \boxed{6}$$

6 To find $\sum_{n=1}^{\infty} \frac{1}{n} \cdot \left(\frac{1}{3}\right)^n$ first find $f(x) = \sum_{n=1}^{\infty} \frac{1}{n} \cdot x^n$ as a function of x (where it converges).

Since $\frac{1}{n} x^n = \int_0^x t^{n-1}\,dt$, we can write

$$f(x) = \int_0^x \left(\sum_{n=1}^{\infty} t^{n-1} \right) dt$$

$$= \int_0^x \left(\frac{1}{1-t} \right) dt \qquad \left(\text{since } \sum_{n=1}^{\infty} t^{n-1} \text{ is a geo. series with first term 1 and common ratio } t \right)$$

$$= \Big[-\ln|1-t| \Big]_0^x$$

$$= -\ln|1-x| = \ln\left|\frac{1}{1-x}\right|$$

Taking $x = 1/3$, we obtain $\sum_{n=1}^{\infty} \frac{1}{n} \cdot \left(\frac{1}{3}\right)^n = \ln\left|\frac{1}{1-1/3}\right| = \ln(3/2)$

$$= \boxed{\ln 3 - \ln 2}$$

7

We saw in class that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\Rightarrow \quad \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot x^{2n}$$

$$\Rightarrow \quad \tan^{-1} x = \int_0^x \left(\sum_{n=0}^{\infty} (-1)^n t^{2n} \right) dt$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{2n+1} x^{2n+1}$$ (where this converges)

so assuming this series converges at $x=1$, (it does, as we'll see later)

$$\frac{\pi}{4} = \tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{2n+1}$$

$$\Rightarrow \quad \boxed{\pi = \sum_{n=0}^{\infty} \frac{(-1)^n 4}{2n+1}}$$

or $\pi = \frac{4}{1} - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \cdots$

(unfortunately this converges rather slowly, so different series would be used in practice).

(8)

From problem #7:

$$\tan^{-1}x = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{2n+1} .$$

So assuming convergence,

$$\boxed{\tan^{-1}\left(\tfrac{1}{3}\right) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{3^{2n+1} \cdot (2n+1)}}$$

$$= \frac{1}{3 \cdot 1} - \frac{1}{\cancel{3}\, 3^3 \cdot 3} + \frac{1}{3^5 \cdot 5} - \frac{1}{3^7 \cdot 7} + \cdots$$

$$(\approx 0.322)$$

(this series converges rather fast).

(9) & (10) <u>Postponed</u> to PSet 8.

(11)

a) Note that $f(x) = \frac{1}{\sqrt{x}}$ is positive & decreasing, so we can apply the integral test.

$$\int_1^{\infty} \frac{1}{\sqrt{x}}\,dx = \lim_{b\to\infty}\left[2\sqrt{x}\,\right]_1^b = \left(\lim_{b\to\infty} 2\sqrt{b}\right) - 2$$

$$= \infty \text{ (diverges)}.$$

So $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges as well.

b)

$$\int_1^\infty \frac{1}{x^7}\,dx = \left[-\frac{1}{6}\cdot\frac{1}{x^6}\right]_1^\infty = 1/6 \quad \text{(converges)}.$$

since $\frac{1}{x^7}$ is positive & decreasing, the integral test applies,

so $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^7}$ <u>converges</u> as well.

c) $\displaystyle\int \frac{1}{\ell(\ln \ell)^2}\,d\ell$ $\quad u = \ln \ell$, $du = \frac{1}{\ell}\,d\ell$

$$= \int \frac{1}{u^2}\,du$$

$$= -\frac{1}{u} + C = -\frac{1}{\ln(\ell)} + C$$

so $\displaystyle\int_{10}^\infty \frac{1}{\ell\cdot(\ln \ell)^2}\,d\ell = \left[-\frac{1}{\ln(\ell)}\right]_{10}^\infty = \frac{1}{\ln 10}$ (converges)

since $\frac{1}{\ell\cdot(\ln \ell)^2}$ is positive & decreasing,

$\displaystyle\sum_{\ell=10}^{\infty} \frac{1}{\ell\cdot(\ln \ell)^2}$ <u>converges</u>, by the integral test.

d) $\displaystyle\int \frac{1}{x\ln x}\,dx$ $\quad u = \ln x$, $du = \frac{1}{x}\,dx$

$$= \int \frac{1}{u}\,du = \ln|u| + C$$

$$= \ln|\ln x| + C$$

so $\displaystyle\int_{20}^\infty \frac{1}{x\ln x}\,dx = \left[\ln|\ln x|\right]_{20}^\infty = \infty$ (diverges)

$\frac{1}{x\ln x}$ is positive & decreasing (for $x>1$), so the integral test applies: $\displaystyle\sum_{\ell=20}^{\infty} \frac{1}{\ell\cdot\ln \ell}$ <u>diverges</u> as well.

(12)

$$\int_0^\infty x^n e^{-x}\,dx$$

$$= \lim_{b\to\infty} \int_0^b x^n e^{-x}\,dx \qquad \begin{array}{ll} u = x^n & dv = e^{-x}dx \\ du = n\cdot x^{n-1}dx & v = -e^{-x} \end{array}$$

$$= \lim_{b\to\infty} \left[\left[-x^n e^{-x}\right]_0^b + \int_0^b n\cdot x^{n-1}\cdot e^{-x}\,dx \right]$$

$$= \lim_{b\to\infty}\left(-b^n\cdot e^{-b} + 0^n\cdot e^0\right) + \lim_{b\to\infty}\int_0^b n\cdot x^{n-1} e^{-x}\,dx$$

$$= -\lim_{b\to\infty} \frac{b^n}{e^b} + n\cdot\int_0^\infty x^{n-1} e^{-x}\,dx$$

and $\lim_{b\to\infty} \frac{b^n}{e^b} = 0$ by applying l'hôpital's rule n times, or noting that exponential functions always dominate polynomials. So

$$\int_0^\infty x^n e^{-x}\,dx = n\cdot\int_0^\infty x^{n-1} e^{-x}\,dx.$$

13

In all parts, the init. value problem is:

$$1 = Q'' + R\cdot Q' + 100\,Q$$
$$Q(0) = 0$$
$$Q'(0) = 0$$

which can be written

$$0 = u'' + R\cdot u' + 100u$$
$$u(0) = -0.01$$
$$u'(0) = 0$$

where $u(t) = Q(t) - 0.01$

so the char. eqn. is

$$\lambda^2 + R\lambda + 100 = 0.$$

a) $\lambda^2 + 100$ has sol'ns $\lambda = \pm 10i$

So $e^{10it} = \cos(10t) + i\sin(10t)$ is a complex sol'n to the diffEq.

$\Rightarrow$ $u(t) = C\cdot\cos(10t) + D\cdot\sin(10t)$ is gen'l sol'n.

Using the initial conditions,

$$-0.01 = C + D\cdot 0 = C$$
$$0 = -10C\cdot 0 + 10D = 10D$$

so $\boxed{Q(t) = 0.01\cdot(1 - \cos(10t))}$

b)

$\lambda^2 + 16\lambda + 100$ has sol'ns $\lambda = -8 \pm 6i$

$\Rightarrow \quad e^{-8t}\cos(6t) + i\, e^{-8t}\sin(6t)$ is a complex sol'n

$\Rightarrow \quad u(t) = C\cdot e^{-8t}\cos(6t) + D\cdot e^{-8t}\sin(6t)$

$u'(t) = -8Ce^{-8t}\cos(6t) - 6Ce^{-8t}\sin(6t)$
$\quad -8De^{-8t}\sin(6t) + 6De^{-8t}\cos(6t)$

where $\quad -0.01 = u(0) = C$

$0 = u'(0) = -8C + 6D$

$\Rightarrow \quad C = -0.01 = -1/100$

and $D = \frac{4}{3}C = -1/75$

so $\quad \boxed{Q(t) = 0.01 - \frac{1}{100}e^{-8t}\cos(6t) - \frac{1}{75}e^{-8t}\sin(6t)}$

c) $\lambda^2 + 25\lambda + 100 = (\lambda+5)(\lambda+20)$ has sol'ns $-5, -20$

$\Rightarrow \quad u(t) = C\cdot e^{-5t} + D\cdot e^{-20t}$
$u'(t) = -5C\cdot e^{-5t} - 20D\cdot e^{-20t}$

$\Rightarrow \quad -0.01 = C + D$

$0 = -5C - 20D$ ie. $C = -4D$

$\Rightarrow \quad C = -4D$ and $-0.01 = -4D + D = -3D$

$\Rightarrow \quad D = \frac{1}{300}$ and $C = -\frac{1}{75}$

so $\quad \boxed{Q(t) = 0.01 - \frac{1}{75}e^{-5t} + \frac{1}{300}e^{-20t}}$

d) The oscillation stops when the char. eqn. has real roots.
By the quadratic formula, this occurs when

$$R^2 - 4 \cdot 100 \geq 0$$

ie. $$R^2 \geq 400$$

$$R \geq 20.$$ (or $R \leq -20$ if resistance could be negative)

So once R is $\boxed{20 \text{ Ohms}}$ the oscillation disappears.