Math 19 Review Problems   12/10/14

① Find the Taylor series of $\frac{1}{1+2x^2}$ with center $x=0$. What is its radius of convergence?

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$\Rightarrow$ (sub. $-2x^2$ for x) $\frac{1}{1+2x^2} = \sum_{n=0}^{\infty} (-2x^2)^n = \boxed{\sum_{n=0}^{\infty} (-2)^n \cdot x^{2n}}$

Ratio test: $L = \lim_{n\to\infty} \left| \frac{(-2)^{n+1}}{(-2)^n} \cdot \frac{x^{2n+2}}{x^{2n}} \right| = 2 \cdot |x^2|$.

So $L<1 \iff x^2 < \frac{1}{2} \iff |x| < \frac{1}{\sqrt{2}}$; radius of conv. $= \boxed{\frac{1}{\sqrt{2}}}$.

② Find the quadratic approximation of $\sqrt{1+\sin x}$ around $x=0$.

$f(x) = \sqrt{1+\sin x}$ | $f(0) = 1$

$f'(x) = \frac{\cos x}{2\sqrt{1+\sin x}}$ | $f'(0) = \frac{1}{2}$

$f''(x) = \frac{-\sin x}{2\sqrt{1+\sin x}} + \frac{\cos x}{2} \cdot \left(-\frac{1}{2}\right) \cdot \frac{\cos x}{(1+\sin x)^{3/2}}$ | $f''(0) = 0 + \frac{1}{2} \cdot \left(-\frac{1}{2}\right) \cdot \frac{1}{1^{3/2}} = -\frac{1}{4}$

So $P_2(x) = \boxed{1 + \frac{1}{2}x - \frac{1}{8}x^2}$

③ Find a series (of rational numbers) whose sum converges to

$$\int_0^2 \sin(x^2)\,dx = \int_0^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2(2n+1)}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot \left[\frac{1}{4n+3} x^{4n+3}\right]_0^2 = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 2^{4n+3}}{(2n+1)! \cdot (4n+3)}}$$

④ Find the Taylor series of $f(x) = \frac{1}{2}(e^x + e^{-x})$.

$$\frac{1}{2}\left[\sum_{n=0}^{\infty} \frac{1}{n!} x^n + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n\right] = \sum_{n=0}^{\infty} \frac{1+(-1)^n}{2} \cdot \frac{1}{n!} \cdot x^n$$

note $\frac{1+(-1)^n}{2} = \begin{cases} 1 & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$, so this is also $= \boxed{\sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}}$

⑤ Find the Fourier series (2π-periodic) of $f(x)$, where:

$$f(x) = \begin{cases} 1 & -\pi/2 \le x < \pi/2 \\ 0 & -\pi \le x < -\pi/2 \text{ or } \pi/2 \le x < \pi \end{cases}$$

$$f(x+2\pi) = f(x).$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\cdot\pi = 1/2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\cos(nx)\,dx = \frac{1}{n\pi}\left[\sin(nx)\right]_{-\pi/2}^{\pi/2}$$

$$= \frac{2}{n\pi}\cdot\sin\left(\frac{n\pi}{2}\right)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\sin(nx)\,dx = \frac{1}{n\pi}\left[\cos(nx)\right]_{-\pi/2}^{\pi/2} = 0.$$

$$\boxed{\frac{1}{2} + \sum_{n=1}^{\infty}\frac{2}{n\pi}\cdot\sin\left(\frac{n\pi}{2}\right)\cdot\cos(nx)} \text{ or equivalently } \boxed{\frac{1}{2} + \sum_{n=0}^{\infty}\frac{2\cdot(-1)^n}{(2n+1)\pi}\cos((2n+1)x)}$$

⑥ Find the real & complex Fourier coeffs. of $\sin^2 x$.
(hint: find a way to avoid taking any integrals).

$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x)$, which is a Fourier series already.

$\boxed{a_0 = 1/2,\quad a_2 = -1/2,\quad \text{all other real coeffs} = 0.}$

For complex: $\frac{1}{2} - \frac{1}{2}\cos(2x) = \frac{1}{2} - \frac{1}{4}e^{-2ix} - \frac{1}{4}e^{2ix}$

$\Rightarrow$ $\boxed{c_{-2} = -1/4,\quad c_0 = 1/2,\quad c_2 = -1/4 \quad \text{\& others are } 0}$

⑦ Find the steady-state (2π-periodic) solution to

$$f''(t) + 10f(t) = \cos t + \cos(3t)$$

Let $v(t) = \cos t + \cos(3t) = \frac{1}{2}e^{-it} + \frac{1}{2}e^{it} + \frac{1}{2}e^{-3it} + \frac{1}{2}e^{3it}$

Note $c_n(f) = \frac{1}{(in)^2 + 10}\cdot c_n(v) = \frac{1}{10-n^2}\cdot c_n(v)$. Thus:

$$\left.\begin{array}{l} c_{-3}(v) = 1/2 \\ c_{-1}(v) = 1/2 \\ c_1(v) = 1/2 \\ c_3(v) = 1/2 \\ \text{other } c_n(v) = 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} c_{-3}(f) = \frac{1}{2}\cdot\frac{1}{10-(-3)^2} = \frac{1}{2} \\ c_{-1}(f) = \frac{1}{2}\cdot\frac{1}{10-(-1)^2} = \frac{1}{18} \\ c_1(f) = \frac{1}{2}\cdot\frac{1}{10-1^2} = \frac{1}{18} \\ c_3(f) = \frac{1}{2}\cdot\frac{1}{10-3^2} = \frac{1}{2} \\ \text{all other } c_n(f) = 0. \end{array}\right.$$

Hence

$$f(t) = \frac{1}{2}e^{-3it} + \frac{1}{18}e^{-it} + \frac{1}{18}e^{it} + \frac{1}{2}e^{3it}$$

$$= \frac{1}{2}(e^{-3it} + e^{3it}) + \frac{1}{18}(e^{-it} + e^{it})$$

$$= \boxed{\frac{1}{9}\cos t + \cos(3t)}$$