Suggested reading for this week (from the textbook):
§2.3 (Pigeonhole principle), §3.1 (Divisibility)

Study items for PSet 4:

- Know the statement of the "principle of mathematical induction."
- Proof by induction (ordinary form)
- Proof by strong induction
- Proving formulas or inequalities for sums via induction.
- Vocabulary: "base case," "inductive hypothesis," "inductive step."
- How to recognize when a problem is well-suited to an inductive approach.

Problems from the book: (First two numbers refer to the section number)

- 2.2.2(a) and (b) (Formulas for sums of squares and cubes)
- 2.2.9 (Bernoulli's inequality)
- 2.2.15 (A flawed inductive proof that all horses are the same color)

Supplemental problems:

1. Prove that for all $n \in \mathbb{N}$, $6^n - 1$ is a multiple of 5 (i.e. is equal to $5k$ for some integer k).
2. Prove that for all $n \geq 3$, the following inequality holds.
$$n^2 \geq 2n + 1$$
3. We proved the following theorem in class, using strong induction on n: "Every natural number n can be written as a sum (where a single term counts as a sum) of distinct powers of 2." On the last page of this problem set is a flawed proof by ordinary (weak) induction. Identify the flaw in the proof (find the precise sentence that is not valid, and briefly explain the error).
4. (a) Prove that for all natural numbers n, $\frac{n-1}{n^2} \leq \frac{1}{n+1}$. This may be a useful step to apply in part (b). I suggest trying to prove it directly, rather than by induction on n.
   (b) Prove, by induction on n, that for all natural numbers n
$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \geq \frac{3}{2} - \frac{1}{n+1}.$$
5. Compute each the following sums (as in class on Wednesday, we regard a single number as a "sum" with only one term).
$$1$$
$$1 + 3$$
$$1 + 3 + 5$$
$$1 + 3 + 5 + 7$$

From the values you find, formulate a conjecture about the value of such a sum in general. Prove your conjecture by induction.

6. The *Fibonacci sequence* is the following sequence of numbers.

$$1, 1, 2, 3, 5, 8, 13, 21, \cdots$$

In this sequence, each number after the first two is equal to the sum of the previous two. In symbols, we denote these numbers $f_1, f_2, f_3, \cdots$, where $f_1 = 1,\ f_2 = 1$, and $f_{n+2} = f_{n+1} + f_n$ for all $n \in \mathbb{N}$.

Prove that for all $n \in \mathbb{N}$,

$$f_1^2 + f_2^2 + \cdots + f_n^2 = f_n \cdot f_{n+1}.$$

A flawed proof of existence of binary expansion (referred to in Supplement 3)

Theorem: Every natural number n can be written as a sum (where a single term counts as a sum) of distinct powers of 2.

"Proof:" By (ordinary) induction on n.

Base case: Suppose $n = 1$. Then $n = 2^0$, which is a sum (with only one term) of powers of two. So the claim holds for $n = 1$.

Inductive step: Suppose that $n \geq 1$ and the claim holds for n, i.e. n is a sum of distinct powers of 2. We will show that $n + 1$ is also equal to a sum of distinct powers of 2.

By inductive hypothesis,

$$n = 2^{e_1} + 2^{e_2} + \cdots + 2^{e_\ell},$$

for some distinct nonnegative exponents $e_1, \cdots, e_\ell$. Therefore we may write

$$n + 1 = (2^{e_1} + 2^{e_2} + \cdots + 2^{e_\ell}) + 1,$$

and therefore

$$n + 1 = 2^{e_1} + 2^{e_2} + \cdots + 2^{e_\ell} + 2^0.$$

This is a sum of distinct powers of 2. Therefore $n + 1$ is also a sum of distinct powers of 2; this completes the induction.