The problems below come from a variety of sources, including previous final exams. I may add more problems to this list over the next couple days. I will try to post solutions to at least some of them, and can post specific solutions by request.

1. Prove the following "divisibility rules."
   (a) Prove the following famous divisibility rule for 9: a number $n \in \mathbb{N}$ is divisible by 9 if and only if the sum of its digits is divisible by 9.
   (b) Suppose that $n = 10m + d$, so that d is the units digits and m is the part of the number before the units digit. Prove that n is divisibly by 7 if and only if $m - 2d$ is divisible by 7.

2. (From 2014 final exam) Let p be an odd prime other than 5 Prove that $\left(\dfrac{-5}{p}\right) = 1$ if and only if $p \equiv 1, 3, 7, \text{ or } 9 \pmod{20}$.

3. (From 2014 final exam) Let a, b, c be integers. Suppose that p is an odd prime which does not divide a.
   (a) Show the number of solutions to
   $$ax^2 + bx + c \equiv 0 \bmod p$$
   is given by $1 + \left(\frac{b^2-4ac}{p}\right)$ where $\left(\dfrac{\cdot}{p}\right)$ is the Legendre symbol.
   (b) Find all the solutions to $2x^2 + 9x + 10 \equiv 1 \bmod 11$.

4. (From 2014 final exam)
   (a) Conjecture a relationship between $\phi(n)$ and $\phi\left(n^2\right)$.
   (b) Prove your conjecture is true.

5. (From 2014 final exam) Suppose that $p \geq 2$ is a prime number such that $p \equiv 1 \bmod 8$. Compute the following:
   (a) $\left(\frac{p+1}{p}\right)$
   (b) $\left(\frac{p-1}{p}\right)$
   (c) $\left(\frac{(p-1)(p+2)}{p}\right)$

6. (From 2016 final exam) Compute each of the following:
   (a) Find $\sigma(2000)$, where $\sigma(n)$ is the sum of all divisors of n (including 1 and n).
   (b) Find the smallest positive integer which is congruent to -1 modulo 3 and 7 and which is divisible by 11.
   (c) Simplify $7^{167} (\text{mod} 20)$.

7. (From the 2016 final exam) Let p be prime. Prove that the congruence $x^3 + x \equiv 0 \bmod p$ if has exactly three solutions (mod p) if and only if $p \equiv 1 \bmod 4$.

8. (from the 2017 final exam) Compute each of the following.
   (a) The two solutions to the equation $12x + 9y = 15$ with the smallest positive x-values.

(b) The sum of the divisors of 3000 .

(c) The smallest positive integer that leaves a remainder of 2 when divided by 12 , a remainder of -1 when divided by 7 and is divisible by 5 .

(d) A solution to $x^{47} \equiv 11 (\text{mod} 100)$.

9. (from the 2017 final exam) For each of the following, either give a short proof of the statement or give a counter example.
   (a) A number is divisible by 2^n if and only if its last n digits are divisible by 2^n.
   (b) $n^3 + 1$ is composite for every natural number $n \geq 2$.
   (c) If $a = bq + r$ then $\gcd(a, q) = \gcd(q, r)$.
   (d) If $a = bq + r$ then $\gcd(a, b) = \gcd(q, r)$.
   (e) The number $(a - b)(a - c)(b - c)$ is even for all integers a, b, c.
   (f) The equation $x^2 + 3y^4 = 7z^2$ has no nonzero integer solutions.
   (g) The equation $(x + y)^2 = 79 + 2xy$ has no integer solutions.

10. (from the 2017 final exam) Let $p \geq 5$ be a prime. Show that the congruence

$$x^2 - 3y^3 \equiv 0 \pmod p$$

has a nontrivial solution (that is, a solution in which $x, y \not\equiv 0 \bmod p$) if and only if $p \equiv \pm 1 \bmod 12$.

11. (from the 2018 final exam) Let p be an odd prime and let $k \in \mathbb{N}$. Suppose $r \in \mathbb{N}$ is a primitive root for p^k. Prove that either r or $r + p^k$ must be a primitive root for $2p^k$.