Study guide

- (§5) Know the precise definition of $m \mid n$ (m *divides* n)) and the definition of *greatest common divisor* $\gcd(m,n)$.
- (§5) How can you quickly compute $\gcd(m,n)$ by repeatedly taking remainders (the Euclidean algorithm)?
- (terms from class) What do we mean by *integer linear combinations* and the *smallest positive combination* of a and b? How is it related to the greatest common divisor?
- (§6) Given $a, b \in \mathbb{N}$, how can you solve $ax + by = \gcd(a,b)$ for integers x, y using the (extended) Euclidean algorithm?
- (§6) Understand the proof of Theorem 6.1 (the *linear equation theorem*).

Note For the following two problems, please compute the answer *by hand*, showing all of your steps.

1. Use the Euclidean algorithm to find an *integer* solution to the following equation

$$181x + 293y = 1.$$

2. Using the (extended) Euclidean algorithm, find integers x, y that solve each of the following equations (it is enough to produce *one* solution):

   a) $180x + 364y = \gcd(180, 364)$. b) $1001x + 1456y = \gcd(1001, 1456)$

3. (a) Suppose that you have a large supply of 5 dollar coins and a large supply of 6 dollar coins. What is the largest number of dollars that you can *not* make out of some combination of these coins? You do not need to prove your answer, but briefly describe how you have decided upon it.

   (b) Suppose instead that you have 5 dollar and 7 dollar coins. What is the largest amount that you cannot make?

   (c) Suppose instead that you have 5 dollar coins and 8 dollar coins. What is the largest amount that you cannot make?

   (d) Conjecture a formula for the largest amount that you cannot make out of 5 dollar coins and k dollar coins, where k is any integer greater than 1 that is not divisible by 5. You do not need to prove that your formula is correct (but you are encouraged to attempt to do so!).

4. How many ways are there to pay 79 dollars using a combination of five dollar bills and two dollar bills (without receiving any change)?

5. (a) Find integers w, z such that $3w + 20z = 1$.

   (b) Let w be from your answer to the previous part. Find integers x and y such that $6x + 15y = 3w$.

   (c) Put together your previous two answers to give a solution to the equation $6x+15y+20z = 1$, with x, y, z integers.

(d) Follow a similar method to find an integer solution to the equation $155x+341y+385z = 1$.

6. Suppose that a, b are two natural numbers such that $\gcd(a, b) = 1$. Suppose that c is a natural number which is divisible by both a and b. Show that c is divisible by the product ab.

Hint Use the linear equation theorem.

7. The *Fibonacci numbers* are defined as follows: $F_0 = 0,\ F_1 = 1$, and for all $n > 1$,

$$F_n = F_{n-1} + F_{n-2}.$$

For example, the next several Fibonacci numbers are $F_2 = 1,\ F_3 = 2,\ F_4 = 3,\ F_5 = 5,\ F_6 = 8$.

(a) For each value of n from 3 to 9, find integers x_n and y_n such that

$$x_n F_n + y_n F_{n+1} = 1.$$

Arrange these values into a table (you may notice a pattern in this table).

(b) Prove by induction on n that for all $n \geq 0$, $\gcd(F_n, F_{n+1}) = 1$.