Solutions

(a video of me writing these in real time is on Moodle, under "My Media")

Math 271, Linear Algebra, Fall 2022
Midterm 1 Practice Test 1

(This is a modified version of Harris Daniels's Midterm 1 practice test from Fall 2016)

Instructions:

- You may not use notes, books, calculators, cell phones or any other aids.
- You must show all your work to get full credit.
- You have 50 minutes to complete the exam.

Answer the following questions:

1. Let $V = \mathbb{R}^+$ be the vector space whose objects are the positive real numbers with addition and scalar multiplication operations defined by

$$x \oplus y = \underline{xy}, \quad c \odot x = \underline{x^c}.$$

(a) Prove that 1 is the additive identity for the vector space V.

Recall
Add. id.: $\vec{0} \in V$ s.t. $\forall \vec{x} \in V$, $\vec{x} \oplus \vec{0} = \vec{x}$

$\forall x \in \mathbb{R}^+$, $1 \oplus x = 1 \cdot x$
$= x$

Therefore 1 is the add. identity

(b) Prove one of the two distributive laws for the vector space V.

$\forall c \in \mathbb{R}$, $x, y \in \mathbb{R}^+$

$$\underline{c \odot (x \oplus y)} = c \odot (xy) = (xy)^c = x^c y^c = (c \odot x)(c \odot y) = \underline{(c \odot x) \oplus (c \odot y)}$$

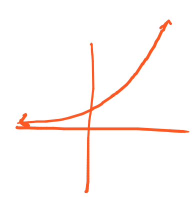
$\forall c, d \in \mathbb{R}$, $x \in \mathbb{R}^+$,

$$\underline{(c+d) \odot x} = x^{c+d} = x^c \cdot x^d = (c \odot x)(c \odot y) = \underline{c \odot x \oplus c \odot y}$$

(c) Prove $V = \text{span}(\{e\})$ where e is the base of the natural logarithm function $\ln(x)$.

$V = \mathbb{R}^+$; $e = 2.71\ldots$

$$\text{Span}\{e\} = \{c \odot e : c \in \mathbb{R}\} = \{e^c : c \in \mathbb{R}\}$$
$= \mathbb{R}^+$, because the range of the function e^x is all of $\mathbb{R}^+$.



2. Is the vector $(4, 0, 6, 9)$ in the span of the set $\{(2, 1, 0, 0), (0, 1, 0, 0), (0, 1, -2, -3)\}$? Justify your answer.

SCRATCH we want to know: are there $x, y, z \in \mathbb{R}$ such that

$$(4,0,6,9) = x\cdot(2,1,0,0) + y\cdot(0,1,0,0) + z\cdot(0,1,-2,-3)\ ?$$
$$= (2x,\ x+y+z,\ -2z,\ -3z)$$

ie. is there a solution to:

$$\begin{cases} 2x = 4 \\ x+y+z = 0 \\ -2z = 6 \\ -3z = 9 \end{cases} \quad ?$$

Observe that solving this gives $z = \frac{9}{-3} = -3$ (from 4^{th} eq'n),
$x = \frac{4}{2} = 2$ (from 1^{st} eq'n),

and $x+y+z=0$ gives $2+y+(-3)=0$, ie. $\underline{y=1}$.

(third eq'n just gives $z = \frac{6}{-2} = -3$ again).

$\Rightarrow$ the numbers $x=2,\ y=1,\ z=-3$ solve this system!

SOLUTION Yes, it is in the span, because

$$2\cdot(2,1,0,0) + 1\cdot(0,1,0,0) - 3(0,1,-2,-3)$$ Linear comb. of the given set

$$= (4,\ 2+1-3,\ 6,\ 9)$$
$$= (4,\ 0,\ 6,\ 9).$$

(!) to prove lin. dep., it's enough to write an example!

3. Put the following linear system into echelon form and use your answer to write down an expression for the solution set in terms of free variables

$$\begin{aligned} x_2 + 2x_3 - x_4 + x_5 &= 1 \\ x_1 \qquad + 4x_3 \qquad + x_5 &= 2 \\ x_1 - 2x_2 \qquad\qquad\qquad &= 0 \end{aligned}$$

$$\left(\begin{array}{ccccc|c} 0 & 1 & 2 & -1 & 1 & 1 \\ 1 & 0 & 4 & 0 & 1 & 2 \\ 1 & -2 & 0 & 0 & 0 & 0 \\ -1 & -0 & -4 & -0 & -1 & -2 \end{array}\right) -R2$$

$$\downarrow$$

$$\left(\begin{array}{ccccc|c} 1 & 0 & 4 & 0 & 1 & 2 \\ 0 & 1 & 2 & -1 & 1 & 1 \\ 0 & -2 & -4 & 0 & -1 & -2 \\ & +2 & +4 & -2 & +2 & +2 \end{array}\right) +2R2$$

$$\downarrow$$

$$\left(\begin{array}{ccccc|c} 1 & 0 & 4 & 0 & 1 & 2 \\ 0 & 1 & 2 & -1+1 & 1-\frac{1}{2} & 1-0 \\ 0 & 0 & 0 & -2/(-2) & 1/(-2) & 0/(-2) \end{array}\right) -\tfrac{1}{2}R3 \quad /(-2)$$

$$\downarrow$$

$$\left(\begin{array}{ccccc|c} 1 & 0 & 4 & 0 & 1 & 2 \\ 0 & 1 & 2 & 0 & 1/2 & 1 \\ 0 & 0 & 0 & 1 & -1/2 & 0 \end{array}\right)$$

$$\begin{cases} x_1 + 4x_3 + x_5 = 2 \\ x_2 + 2x_3 + \frac{1}{2}x_5 = 1 \\ x_4 - \frac{1}{2}x_5 = 0 \end{cases}$$

$$\boxed{\begin{aligned} x_1 &= 2 - 4x_3 - x_5 \\ x_2 &= 1 - 2x_3 - \tfrac{1}{2}x_5 \\ x_4 &= \tfrac{1}{2}x_5 \\ & x_3, x_5 \text{ free.} \end{aligned}}$$

To check this: try plugging in $x_3 = x_4 = 0$ & $x_3 = 1$, $x_4 = 0$ etc.
make sure what you get solves the system!

4. Let V be a vector space and suppose $\{\mathbf{u}, \mathbf{v}\}$ is a basis for V. Prove that $\{2\mathbf{u}-\mathbf{v}, \mathbf{u}+\mathbf{v}\}$ is also a basis for V.

S = $\{2\mathbf{u}-\mathbf{v}, \mathbf{u}+\mathbf{v}\}$

basis: L.I. & span V.

1) S is linearly indep

Suppose $c_1(2\vec{u}-\vec{v}) + c_2(\vec{u}+\vec{v}) = \vec{0}$.

$\Rightarrow (2c_1+c_2)\vec{u} + (-c_1+c_2)\vec{v} = \vec{0}$.

Because $\{\vec{u},\vec{v}\}$ is a basis, it is linearly indep., so

$$\left.\begin{array}{l} 2c_1+c_2=0 \\ \& \ -c_1+c_2=0. \end{array}\right\} \Rightarrow \begin{array}{l} c_1=c_2 \quad \text{(2nd eq'n)} \\ \& \ 2c_1+c_1=0 \\ \Rightarrow 3c_1=0 \\ \Rightarrow c_1=0 \ \& \ c_2=c_1=0. \end{array}$$

Hence $c_1=c_2=0$.

Thus $\{2\vec{u}-\vec{v}, \vec{u}+\vec{v}\}$ is lin. indep.

2) S spans V. (i.e. $\operatorname{Span} S = V$).

"$\subseteq$" since $\vec{u},\vec{v} \in V$, also $2\vec{u}-\vec{v}, \vec{u}+\vec{v} \in V$, so all LC's of them are in V (V is closed under $+$ and scalar $\cdot$). So $\operatorname{Span} S \subseteq V$.

"$\supseteq$" Use the fact that $\{\vec{u},\vec{v}\}$ is a basis... so $V = \operatorname{Span}\{\vec{u},\vec{v}\}$.

For any $\vec{x} \in V$, $\exists\, c_1, c_2$ s.t. $\vec{x} = c_1\vec{u} + c_2\vec{v}$.

I want to rewrite this as a LC of $\{2\vec{u}-\vec{v}, \vec{u}+\vec{v}\}$.. how?

↳ write $\vec{u},\vec{v}$ as LC's of $\{2\vec{u}-\vec{v}, \vec{u}+\vec{v}\}$.

Observe: $\frac{1}{3}\left[(2\vec{u}-\vec{v}) + (\vec{u}+\vec{v})\right] = \vec{u}$

$\frac{1}{3}\left[2(\vec{u}+\vec{v}) - (2\vec{u}-\vec{v})\right] = \vec{v}$

Therefore: $\vec{x} = c_1 \cdot \frac{1}{3}\left[(2\vec{u}-\vec{v}) + (\vec{u}+\vec{v})\right] + c_2 \cdot \frac{1}{3} \cdot \left[2(\vec{u}+\vec{v}) - (2\vec{u}-\vec{v})\right]$

$$\Rightarrow \vec{x} = \underbrace{\left(\tfrac{1}{3}c_1 - \tfrac{1}{2}c_2\right)}_{\in \mathbb{R}} (2\vec{u}-\vec{v}) + \underbrace{\left(\tfrac{1}{3}c_1 + \tfrac{2}{3}c_2\right)}_{\in \mathbb{R}} (\vec{u}+\vec{v})$$

this is a LC of S, hence $\vec{x} \in \operatorname{Span} S$.

Therefore $\operatorname{Span} S = V$.

this should now be fixed in the pdf.

Define a

5. ~~Find a basis for the~~ subspace of $\underline{P_2(\mathbb{R})}$ given by $W = \{f \in P_2(\mathbb{R}) \mid \underline{f'(2) = 0}\}$. Find a set that spans W. (Note: f' here refers to the derivative of f.)

$$P_2(\mathbb{R}) = \{a_0 + a_1 x + a_2 x^2 : a_0, a_1, a_2 \in \mathbb{R}\}$$

homog. lin. eqn

$$W = \{\underbrace{a_0 + a_1 x + a_2 x^2}_{f(x)} : \underbrace{a_1 + 2 \cdot a_2 \cdot 2}_{\text{"}f'(2)\text{"}} = 0\}$$

$$f'(x) = a_1 + 2a_2 \cdot x$$

$$\{a_1 + 4a_2 = 0 \iff \underline{a_1 = -4a_2,}$$
$$a_0, a_2 \text{ free.}$$

Linear system w/ just one eqn. in "echelon form"

$$\Rightarrow \quad W = \{a_0 - 4a_2 x + a_2 x^2 : a_0, a_2 \in \mathbb{R}\}$$
$$= \{\underbrace{a_0 \cdot 1 + a_2(-4x + x^2)}_{LC} : a_0, a_2 \in \mathbb{R}\}$$
$$= \operatorname{Span}\{1, -4x + x^2\}.$$

So $\{1, -4x + x^2\}$ is such a set.