1. **True or False:** (No justification necessary.)
   
   (a) [3 points] There exists a set of three vectors \( \{ \vec{u}, \vec{v}, \vec{w} \} \subseteq \mathbb{R}^2 \) that is linearly independent.  
   
   \( \text{T} \) \( \text{F} \)

   \((\text{can't have } n \text{ vectors, lin. indep. in } \mathbb{R}^n)\)

   (b) [3 points] Every system of linear equations has at least one solution.  

   \( \text{T} \) \( \text{F} \)

   \[\begin{align*}
   \text{E.g.} & \quad x + y = 2 \\
   & \quad 2x + 2y = 5
   \end{align*}\]

   (c) [3 points] The set of solutions to a system of linear equations in \( n \) unknowns is a \( \text{T} \) \( \text{F} \) subspace of \( \mathbb{R}^n \).

   \( \Theta \)

   \( (1, 0) \) is a soln, but \( 2 \cdot (1, 0) \) is not

   (d) [3 points] The set \( \{ f \in C^1(\mathbb{R}) \mid f + f' = 0 \} \) is a subspace of \( C^1(\mathbb{R}) \).  

   \( \text{T} \) \( \text{F} \)

   \[\begin{align*}
   C^1(\mathbb{R}) &= \text{differentiable function.} \\
   \text{nonempty?} & \quad f(x) = 0 \quad \text{a such function.} \\
   \text{closure?} & \quad \text{Suppose } f + f' = 0 \\
   & \quad g + g' = 0 \\
   & \quad c \in \mathbb{R}.
   \end{align*}\]

   \[\begin{align*}
   (f + cg) + (f + cg)' &= f + cg + f' + cg' \\
   &= (f + f') + c(g + g') = 0 + 0 = 0.
   \end{align*}\]
2. Let $V$ be a vector space, and let $S \subseteq V$. Define the following terms and phrases. You may use other standard terms without defining them.

(a) [5 points] The set $S$ spans $V$.

\[
\text{Span } S = V.
\]

In other words, $V$ is equal to the set of all linear combinations of vectors in $S$.

\[
V = \{ c_1 \vec{v}_1 + \cdots + c_n \vec{v}_n : c_1, \ldots, c_n \in \mathbb{R} , \quad \vec{v}_1, \ldots, \vec{v}_n \in S \}
\]

(b) [5 points] The set $S$ is linearly independent.

For distinct vectors $\vec{v}_1, \ldots, \vec{v}_n \in S$,
the only choice of constants $c_1, \ldots, c_n \in \mathbb{R}$ such that
\[
C_1 \vec{v}_1 + \cdots + C_n \vec{v}_n = \vec{0}
\]

is $c_1 = c_2 = \cdots = c_n = 0$.

(c) [5 points] The set $S$ is a basis of $V$.

$S$ is lin. indep. & spans $V$. 
3. [15 points] Find a set of vectors spanning the set of solutions to the following system of equations.

\[
\begin{cases}
    x_1 + 2x_2 - x_4 = 0 \\
    -2x_1 - 3x_2 + 4x_3 - 5x_4 = 0 \\
    2x_1 + 4x_2 - 2x_4 = 0
\end{cases}
\]

\[
\begin{bmatrix}
    1 & 2 & 0 & -1 \\
    -2 & -3 & 4 & -5 \\
    2 & 4 & 0 & -2
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
    x_4
\end{bmatrix}
= \begin{bmatrix}
    0 \\
    0 \\
    0
\end{bmatrix}
\]

General soln:

\[
x_1 = 8x_3 - 13x_4 \\
x_2 = -4x_3 + 7x_4 \\
x_3, x_4 \text{ free}
\]

ie. as a vector:

\[
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
    x_4
\end{bmatrix}
= \begin{bmatrix}
    8 & -13 \\
    -4 & 7
\end{bmatrix}
\begin{bmatrix}
    x_3 \\
    x_4
\end{bmatrix}
\]

So set of solutions in \( \text{Span}\{ (8, -4, 1, 0), (-13, 7, 0, 1) \} \)

\[
\{ (8, -4, 1, 0), (-13, 7, 0, 1) \} \quad \text{2-dim set of solutions.}
\]

suggest: check that there are solns!
4. (a) [5 points] Let \( S_1 = \{ (1,1,1), (1,1,0), (1,0,0) \} \subseteq \mathbb{R}^3 \). Is \( S_1 \) linearly independent? Why or why not?

Yes. Suppose that \( c_1, c_2, c_3 \) satisfy

\[ c_1 (1,1,1) + c_2 (1,1,0) + c_3 (1,0,0) = (0,0,0). \]

Then:

\[ \begin{cases} 
  c_1 + c_2 + c_3 = 0 \\
  c_1 + c_2 = 0 \\
  c_1 = 0 
\end{cases} \]

\( \Rightarrow \) (reading bottom-to-top)

\[ c_1 = 0, \quad \text{so} \quad c_1 + c_2 = 0 \Rightarrow c_2 = 0, \]

\[ \text{so} \quad c_1 + c_2 + c_3 = 0 \Rightarrow c_3 = 0. \]

Hence \( c_1 = c_2 = c_3 = 0 \).

Therefore \( S_1 \) is linearly independent.

(b) [5 points] Let \( S_2 = \{ 1 + x + x^2, 2 - x, 3 - 2x + x^2, x - 2x^2 \} \subseteq P_2(\mathbb{R}) \). Is \( S_2 \) linearly independent? Why or why not?

Linearly dependent, because:

\[ c_1 (1 + x + x^2) + c_2 (2 - x) + c_3 (3 - 2x + x^2) + c_4 (x - 2x^2) = 0 \]

\( \Rightarrow \) \( (c_1 + 2c_2 + 3c_3) + (c_1 - c_2 - 2c_3 + c_4) x + (c_1 + c_3 - 2c_4) x^2 = 0 \)

\( \Rightarrow \) \( \begin{cases} 
  c_1 + 2c_2 + 3c_3 = 0 \\
  c_1 - c_2 - 2c_3 + c_4 = 0 \\
  c_1 + c_3 - 2c_4 = 0 
\end{cases} \)

This is a homogenous system of 3 eqns in 4 variables. Since \( 4 > 3 \), a thin row can say it must have a nontrivial solution.

Hence \( \exists \) a nontrivial LC of these polyx that equals 0, i.e. \( S_2 \) is linearly dependent.

\( \text{eg.} \) let \( c_4 = 1 \).

\( c_1 = 0, \ c_2 = -3, \ c_3 = 2, \ c_4 = 1 \)

\( S_2 \) indeed

\[ 0 \cdot (1 + x + x^2) - 3 \cdot (2 - x) + 2 \cdot (3 - 2x + x^2) + 1 \cdot (x - 2x^2) = 0 \]

in a linear dependence.
5. [15 points] Let \( S = \{1 + 2x + 3x^2, 1 + x^2 + x^3, x^2 + x^3\} \subseteq P_3(\mathbb{R}) \). Is \( 3 + 2x + 4x^2 + x^3 \in \text{Span}(S)\)? Justify your answer.

We want to know: do there exist \( c_1, c_2, c_3 \) such that

\[
c_1(1 + 2x + 3x^2) + c_2(1 + x^2 + x^3) + c_3(x^2 + x^3) = 3 + 2x + 4x^2 + x^3
\]

\[\iff (c_1 + c_2) + (2c_1)x + (3c_1 + c_2 + c_3)x^2 + (c_2 + c_3)x^3 = 3 + 2x + 4x^2 + x^3\]

\[\iff \begin{cases} c_1 + c_2 &= 3 \\
2c_1 &= 2 \\
3c_1 + c_2 + c_3 &= 4 \\
c_2 + c_3 &= 1 \end{cases}\]

\[\iff \begin{cases} c_1 = 1 \quad \text{(1st eqn)} \\
c_2 = 3 - c_1 = 2 \quad \text{(2nd eqn)} \\
3 - 1 + 2 + c_3 = 4 \quad \& \quad 2 + c_3 = 1 \quad \text{(3rd & 4th eqns)}.
\end{cases}\]

Both solve to \( c_3 = -1 \).

\[\iff c_1 = 1, \quad c_2 = 2, \quad c_3 = -1.\]

**SOLN** Yes, because

\[
1 \cdot (1 + 2x + 3x^2) + 2 \cdot (1 + x^2 + x^3) - 1 \cdot (x^2 + x^3) = 3 + 2x + 4x^2 + x^3.
\]

Thus a \textbf{a L of S} desired vector.
6. [15 points] Let $V$ be a vector space and let $S$ and $T$ be subsets of $V$. Show that if $\text{Span}(S) = V$ and $S \subseteq \text{Span}(T)$, then $\text{Span}(T) = V$.

"\subseteq" $T \subseteq V$ & $V$ is closed under + & scalar \cdot; so $V$
contains linear combs. of $T$, i.e. $\text{Span}(T) \subseteq V$.

"\supseteq" Suppose $x \in V$.

Then $x \in \text{Span}(S)$ by assumption,

so $x = c_1 \bar{v}_1 + \ldots + c_n \bar{v}_n$ for some $c_1, \ldots, c_n \in \mathbb{R}$ & $\bar{v}_1, \ldots, \bar{v}_n \in S$.

Observe each $\bar{v}_1, \ldots, \bar{v}_n \in \text{Span}(T)$ since $S \subseteq \text{Span}(T)$
by assumption.

$\text{Span}(T)$ is a subspace of $V$, so it is closed
under + and scalar \cdot.

so $c_1 \bar{v}_1 + \ldots + c_n \bar{v}_n \in \text{Span}(T)$,

i.e. $x \in \text{Span}(T)$.

So $V \subseteq \text{Span}(T)$ as desired.

$V = \text{Span}(T)$