Solutions

(This is a modified version of Harris Daniels's Midterm 1 practice test from Fall 2018)

1. True or False: (No justification necessary.)

(a) [3 points] There exists a set of three vectors $\underbrace{\{\vec{u}, \vec{v}, \vec{w}\}}_{3 \text{ vectors}} \subseteq \mathbb{R}^2$ that is linearly independent. **T** **F**

(can't have $> n$ vectors, lin. indep. in $\mathbb{R}^n$)

(b) [3 points] Every system of linear equations has at least one solution. **T** **F**

eg. $x+y=2$

$2x+2y=5$

(c) [3 points] The set of solutions to a system of linear equations in n unknowns is a subspace of $\mathbb{R}^n$. **T** **F**

eg. $x+y=1$

$(1,0)$ is a sol'n, but $2\cdot(1,0)$ is not

(d) [3 points] The set $\{f \in C^1(\mathbb{R}) \mid f + f' = 0\}$ is a subspace of $C^1(\mathbb{R})$. **T** **F**

$C^1(\mathbb{R})$ = differentiable functions.

nonempty? $f(x)=0$ is such a function. ✓

closure? suppose $f+f'=0$

$g+g'=0$

$c \in \mathbb{R}$.

$(f+cg)+(f+cg)' = f+cg+f'+cg'$

$= (f+f')+c(g+g') = 0+0=0.$ ✓

2. Let V be a vector space, and let $S \subseteq V$. Define the following terms and phrases. You may use other standard terms without defining them.

(a) [5 points] The set "S spans V."

Span $S = V$.

In other words, V is equal to the set of all linear combinations of vectors in S.

OR

$$V = \{ c_1\vec{v}_1 + \cdots + c_n\vec{v}_n : c_1, \cdots, c_n \in \mathbb{R}, \quad \vec{v}_1, \cdots, \vec{v}_n \in S \}$$

(b) [5 points] The set S is linearly independent.

For distinct vectors $\vec{v}_1, \cdots, \vec{v}_n \in S$,
the only choice of constants $c_1, \cdots, c_n \in \mathbb{R}$
such that

$$c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0}$$

is $c_1 = c_2 = \cdots = c_n = 0$.

(c) [5 points] The set S is a basis of V.

S is lin. indep. & spans V.

3. [15 points] Find a set of vectors spanning the set of solutions to the following system of equations.

$$\begin{cases} x_1 + 2x_2 \qquad\quad - x_4 = 0 \\ -2x_1 - 3x_2 + 4x_3 - 5x_4 = 0 \\ 2x_1 + 4x_2 \qquad\quad - 2x_4 = 0 \end{cases}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 0 \\ -2_{+2} & -3_{+4} & 4_{+0} & -5_{-2} & 0_{+0} \\ 2_{-2} & 4_{-4} & 0_{-0} & -2_{+2} & 0_{-0} \end{array}\right) \begin{array}{l} \\ +2R1 \\ -2R1 \end{array}$$

$$\downarrow$$

$$\left(\begin{array}{cccc|c} 1 & 2^{-2} & 0^{-8} & -1^{+14} & 0^{-0} \\ 0 & 1 & 4 & -7 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} -2R2 \\ \\ \\ \end{array}$$

$$\downarrow$$

$$\left(\begin{array}{cccc|c} 1 & 0 & -8 & 13 & 0 \\ 0 & 1 & 4 & -7 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

x_1, x_2 bound $\qquad x_3, x_4$ free

General sol'n: $x_1 = 8x_3 - 13x_4$

$x_2 = -4x_3 + 7x_4$

x_3, x_4 free.

ie. as a vector: $(8x_3 - 13x_4, -4x_3 + 7x_4, x_3, x_4)$

$= x_3 \cdot (8, -4, 1, 0) + x_4 (-13, 7, 0, 1)$.

So set of solutions is $\mathrm{Span}\{(8,-4,1,0), (-13,7,0,1)\}$

$$\boxed{\{(8,-4,1,0), (-13,7,0,1)\}}$$

← 2-dim'l set of solutions.

suggest: check that these are sol'ns!

4. (a) [5 points] Let $S_1 = \{(1,1,1),(1,1,0),(1,0,0)\} \subseteq \mathbb{R}^3$. Is S_1 linearly independent? Why or why not?

Yes. Suppose that c_1, c_2, c_3 satisfy

$$c_1(1,1,1) + c_2(1,1,0) + c_3(1,0,0) = (0,0,0).$$

Then: $(c_1+c_2+c_3,\ c_1+c_2,\ c_1) = (0,0,0)$

$$\iff \begin{cases} c_1+c_2+c_3=0 \\ c_1+c_2 =0 \\ c_1 =0 \end{cases}$$

$\Rightarrow$ (reading bottom-to-top)

$c_1 = 0$, so $c_1+c_2=0 \Rightarrow c_2=0$,

so $c_1+c_2+c_3=0 \Rightarrow c_3=0$

Hence $c_1=c_2=c_3=0$.

Therefore S_1 is linearly independent.

(b) [5 points] Let $S_2 = \{1+x+x^2, 2-x, 3-2x+x^2, x-2x^2\} \subseteq P_2(\mathbb{R})$. Is S_2 linearly independent? Why or why not?

Linearly dependent, because:

$$c_1(1+x+x^2) + c_2(2-x) + c_3(3-2x+x^2) + c_4(x-2x^2) = 0$$

$$\iff (c_1+2c_2+3c_3) + (c_1-c_2-2c_3+c_4)x + (c_1+c_3-2c_4)x^2 = 0$$

$$\iff \begin{cases} c_1+2c_2+3c_3 = 0 \\ c_1-c_2-2c_3+c_4=0 \\ c_1+c_3-2c_4=0 \end{cases}$$

this is a homog. system of 3 eq'ns in 4 variables. Since $4>3$, a thm from class says it must have a nontrivial solution!

Hence $\exists$ a nontrivial LC of these polys. that equals 0, i.e. S_2 is linearly dependent.

$$\begin{array}{c} \\ -R1 \\ -R1 \end{array} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 0 & 0 \\ 1 & -1 & -2 & 1 & 0 \\ 1 & 0 & 1 & -2 & 0 \end{array}\right)$$

$$\begin{array}{c} \\ \\ *(-\tfrac{1}{2}) \end{array} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 0 & 0 \\ 0 & -3 & -5 & 1 & 0 \\ 0 & -2 & -2 & -2 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & -3 & -5 & 1 & 0 \end{array}\right) \begin{array}{c} -2R2 \\ \\ +3R2 \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & -2 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & -2 & 4 & 0 \end{array}\right) \begin{array}{c} \\ \\ *(-\tfrac{1}{2}) \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & -2 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 \end{array}\right) \begin{array}{c} -R3 \\ -R3 \\ \\ \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & -2 & 0 \end{array}\right)$$

$c_1 = 0$
$c_2 = -3c_4$
$c_3 = 2c_4$
c_4 free.

e.g. let $c_4 = 1$.
$c_1 = 0,\ c_2 = -3,\ c_3 = 2,\ c_4 = 1.$

& indeed $0\cdot(1+x+x^2) - 3(2-x) + 2(3-2x+x^2) + 1(x-2x^2) = 0$ is a linear dependence.

Alt. sol'n: just write this example

5. [15 points] Let $S = \{1+2x+3x^2, 1+x^2+x^3, x^2+x^3\} \subseteq P_3(\mathbb{R})$. Is $3+2x+4x^2+x^3 \in \mathrm{Span}(S)$? Justify your answer.

SCRATCH

Still show somewhere!

We want to know: do there exist c_1, c_2, c_3 s.t.

$$c_1(1+2x+3x^2) + c_2(1+x^2+x^3) + c_3(x^2+x^3) = 3+2x+4x^2+x^3$$

$$\Leftrightarrow (c_1+c_2) + (2c_1)x + (3c_1+c_2+c_3)x^2 + (c_2+c_3)x^3 = 3+2x+4x^2+x^3$$

$$\Leftrightarrow \begin{cases} c_1 + c_2 = 3 \\ 2c_1 = 2 \\ 3c_1 + c_2 + c_3 = 4 \\ c_2 + c_3 = 1 \end{cases}$$

$\Leftrightarrow$ $c_1 = 1$ (2^{nd} eq'n), $c_2 = 3 - c_1 = 2$ (1^{st} eq'n),

& $\underbrace{3\cdot 1 + 2 + c_3 = 4}$ & $\underbrace{2 + c_3 = 1}$ (3^{rd} & 4^{th} eq'ns).

both solve to $\underline{c_3 = -1}$.

$\Leftrightarrow$ $c_1 = 1$, $c_2 = 2$, $c_3 = -1$.

SOLN

Yes, because

$$\underbrace{1\cdot(1+2x+3x^2) + 2\cdot(1+x^2+x^3) - 1\cdot(x^2+x^3)}_{\text{this is a LC of } S} = \underbrace{3+2x+4x^2+x^3}_{\text{desired vector}}.$$

6. [15 points] Let V be a vector space and let S and T be subsets of V. Show that if $\text{Span}(S) = V$ and $S \subseteq \text{Span}(T)$, then $\text{Span}(T) = V$.

"$\subseteq$" $T \subseteq V$ & V is closed under $+$ & scalar $\cdot$; so V contains linear combs. of T, ie. $\text{Span}\,T \subseteq V$.

"$\supseteq$" Suppose $\vec{x} \in V$.

Then $\vec{x} \in \text{Span}\,S$ by assumption

so $\vec{x} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$ for some $c_1, \ldots, c_n \in \mathbb{R}$ & $\vec{v}_1, \ldots, \vec{v}_n \in S$.

Observe each $\vec{v}_1, \ldots, \vec{v}_n \in \text{Span}(T)$ since $S \subseteq \text{Span}(T)$ by assumption.

Span(T) is a subspace of V, so it's closed under $+$ and scalar $\cdot$.

so $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n \in \text{Span}(T)$,

ie. $\vec{x} \in \text{Span}(T)$.

So $V \subseteq \text{Span}(T)$ as desired.

$V = \text{Span}(T)$