Amherst College
Department of Mathematics

Math 271 — Midterm 2 Practice Exam 1 — Spring 2022

Name: Solutions

This is a modified version of a practice exam from Fall 2016.

Read This First!

- Please read each question carefully. Show **ALL** work clearly in the space provided.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable.
- Answers must be clearly labeled in the spaces provided after each question.
- Please cross out or fully erase any work that you do not want graded.
- The point value of each question is indicated after its statement.
- No books or other references are permitted.
- Calculators are not allowed and you must show all your work.

Grading - For Administrative Use Only

Question:	1	2	3	4	5	Total
Points:	20	15	0	10	10	55
Score:						

1. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is linear, $T((1,0)) = (1,2)$, and $T((1,1)) = (3,5)$.

   (a) What is $T((2,3))$? [5]

   scratch: to find these numbers: $c_1(1,0) + c_2(1,1) = (2,3)$ $\Leftrightarrow \begin{cases} c_1 + c_2 = 2 \\ c_2 = 3 \end{cases} \Leftrightarrow \begin{array}{l} c_1 = -1 \\ c_2 = 3 \end{array}$

   $$(2,3) = -1\cdot(1,0) + 3\cdot(1,1)$$

   so since T is linear,

   $$\begin{aligned} T(2,3) &= -T(1,0) + 3T(1,1) \\ &= -(1,2) + 3\cdot(3,5) \\ &= \underline{(8,13)}. \end{aligned}$$

   (b) Is T injective? [5]

Recall: T injective $\Leftrightarrow$ kernel is trivial.

Now, $\forall x, y \in \mathbb{R}$,

$$\begin{aligned} T(x,y) &= T\big((x-y)(1,0) + y\cdot(1,1)\big) \\ &= (x-y)\cdot T(1,0) + y\cdot T(1,1) \\ &= (x-y)\cdot(1,2) + y\cdot(3,5) \\ &= (x-y+3y,\ 2x-2y+5y) \\ &= (x+2y,\ 2x+3y). \end{aligned}$$

Hence, $(x,y) \in \ker T \Leftrightarrow x + 2y = 0$ & $2x + 3y = 0$.

reducing this linear system gives $\underset{-2R1}{} \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 3 & 0 \\ {\scriptstyle -2} & {\scriptstyle -4} & {\scriptstyle -0} \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -1 & 0 \end{array}\right) \begin{array}{l} {\scriptstyle +2R2} \\ {\scriptstyle *(-1)} \end{array} \to \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right)$

so the only solution to this system is $x=y=0$, ie. $\ker T = \{\vec{0}\}$.

This implies that T is injective.

(continued on reverse)

(c) Let α and β be the standard basis for $\mathbb{R}^2$. Compute $[T]_\alpha^\beta$. [10]

As found in part (b),

$$T(x,y) = (x+2y,\ 2x+3y).$$

Written as column vectors,

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+2y \\ 2x+3y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

So, using the standard basis, the matrix representation is $\underline{\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}}$.

Alternative solution:

observe that the first column of $[T]_\alpha^\beta$ is

$$[T\vec{e}_1]_{std} = T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

The second column is $[T\vec{e}_2]_{std} = T\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, which we may obtain as follows:

$$T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = T\begin{pmatrix} 1 \\ 1 \end{pmatrix} - T\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} 3 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

Combining gives the matrix $\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$.

2. Let $V = P_2(\mathbb{R})$, $W = \mathbb{R}^2$, $\alpha = \{1, 1+x, 1+x+x^2\}$, and β is the standard basis for $\mathbb{R}^2$. Suppose that $T : V \to W$ is a linear transformation such that $[T]_\alpha^\beta = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \end{bmatrix}$.

   (a) Find a basis for $\mathrm{Ker}(T)$. [10]

$p \in \ker(T) \iff T(p) = \vec{0} \iff [T(p)]_\beta = \vec{0}$

$\iff [T]_\alpha^\beta [p]_\alpha = \vec{0}$.

Now, letting $[p]_\alpha = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, this matrix equation is the homog. system

$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ ie. $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ \underset{-1}{1} & \underset{-2}{0} & \underset{-3}{1} & 0 \end{array}\right)_{-R1} \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -2 & -2 & 0 \end{array}\right)\begin{matrix} +R1 \\ *(-\frac{1}{2}) \end{matrix}$

$\longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right)$,

which has gen'l solution $x = -z$, $y = -z$, z free, so

$p \in \ker(T) \iff [p]_\alpha = \begin{pmatrix} -z \\ -z \\ z \end{pmatrix} = z\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$. So $\ker(T)$ is spanned

by the poly. with coords. $\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$ in basis α, ie. $-1-(1+x)+(1+x+x^2) = -1+x^2$.

That is. $\{-1+x^2\}$ is a basis for $\ker(T)$.

   (b) Is T surjective? Justify your answer. [5]

By part (a), $\mathrm{nullity}(T) = 1$. By rank-nullity,

$\mathrm{rank}(T) = \dim P_2(\mathbb{R}) - 1 = 3 - 1 = 2$.

Since $\dim \mathbb{R}^2 = 2$, it follows that $\dim \mathbb{R}^2 = \mathrm{rank}(T)$,

so yes, T is surjective.

3. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $T((a_1, a_2, a_3)) = (a_1 + a_2, a_2, a_1 - a_3)$. Show that T is a invertible.

Observe that

$$T(a_1, a_2, a_3) = \vec{0} \iff \begin{cases} a_1 + a_2 = 0 \\ a_2 = 0 \\ a_1 - a_3 = 0 \end{cases}.$$

Solving this system by hand gives $a_2 = 0$ (2nd eq'n), so $a_1 + 0 = 0$ and thus $a_1 = 0$ (1st eq'n), & $0 - a_3 = 0 \Rightarrow a_3 = 0$ (3rd eq'n), so $a_1 = a_2 = a_3 = 0$ is the only solution.

Hence $\ker(T) = \{\vec{0}\}$, ie.

$$\text{nullity}(T) = 0,$$

which implies that T is <u>injective</u>.

By rank-nullity, $\text{rank}(T) = \dim \mathbb{R}^3 - \text{nullity}(T) = 3$, so T is also <u>surjective</u>.

Hence T is invertible.

(alternate ending: cite the "two-out-of-three" theorem).

4. Let $V = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \,|\, x_1 + 2x_2 + 3x_3 = 0\}$. Prove that V is isomorphic to $\mathbb{R}^2$. [10]

Rewriting V in parameterized form,

$$V = \{(-2x_2 - 3x_3, x_2, x_3) : x_2, x_3 \in \mathbb{R} \text{ (free)}\}$$
$$= \text{span}\{(-2,1,0), (-3,0,1)\}.$$

These two vectors are LI, since

$$x_2(-2,1,0) + x_3(-3,0,1) \Rightarrow (-2x_2 - 3x_3, x_2, x_3) = (0,0,0)$$
$$\Rightarrow x_2 = x_3 = 0.$$

so they are a basis for V, & thus $\dim V = 2$.

As proved in class, if $\dim V = n$, then V is isomorphic to $\mathbb{R}^n$, & the desired result follows.

More explicitly, the linear map

$$T : \mathbb{R}^2 \to V$$

given by $T(x,y) = x \cdot (-2,1,0) + y \cdot (-3,0,1)$

is invertible, i.e. an isomorphism

5. Suppose that $T : V \to W$ is a linear transformation such that $\dim \ker T = 0$. Prove that if $\vec{v}_1, \vec{v}_2 \in V$ satisfy $T(\vec{v}_1) = T(\vec{v}_2)$, then $\vec{v}_1 = \vec{v}_2$. [10]

Since $\dim \ker(T) = 0$, $\ker(T) = \{\vec{0}\}$.

If $T(\vec{v}_1) = T(\vec{v}_2)$, then

$$T(\vec{v}_1) - T(\vec{v}_2) = \vec{0},$$

$\Rightarrow$ $T(\vec{v}_1 - \vec{v}_2) = \vec{0}$ (since T is linear)

$\Rightarrow$ $\vec{v}_1 - \vec{v}_2 \in \ker(T)$

$\Rightarrow$ $\vec{v}_1 - \vec{v}_2 = \vec{0}$

$\Rightarrow$ $\vec{v}_1 = \vec{v}_2$.